CSCI 497P/597P: Computer Vision

Projective Geometry

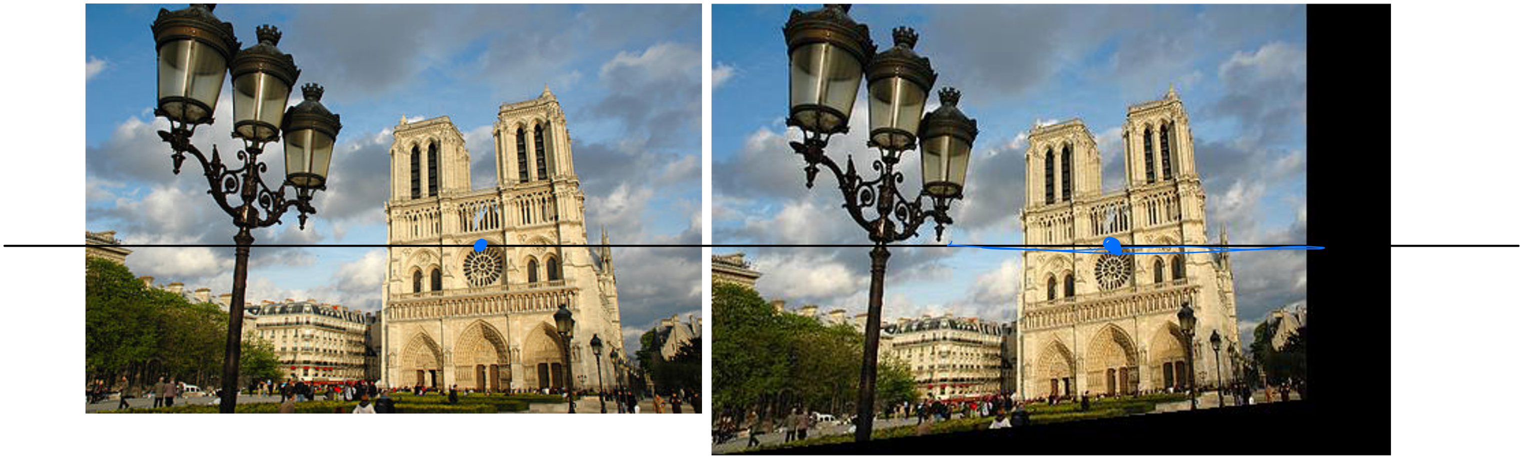
Announcements

- Exam will be out later today, due Monday 10pm.
- Academic honesty:
  - OK – any course materials
    - lecture slides, notes, and videos;
    - your own notes your assignment code, your HW solutions;
    - the textbook
  - Not OK:
    - other people
    - the internet at large

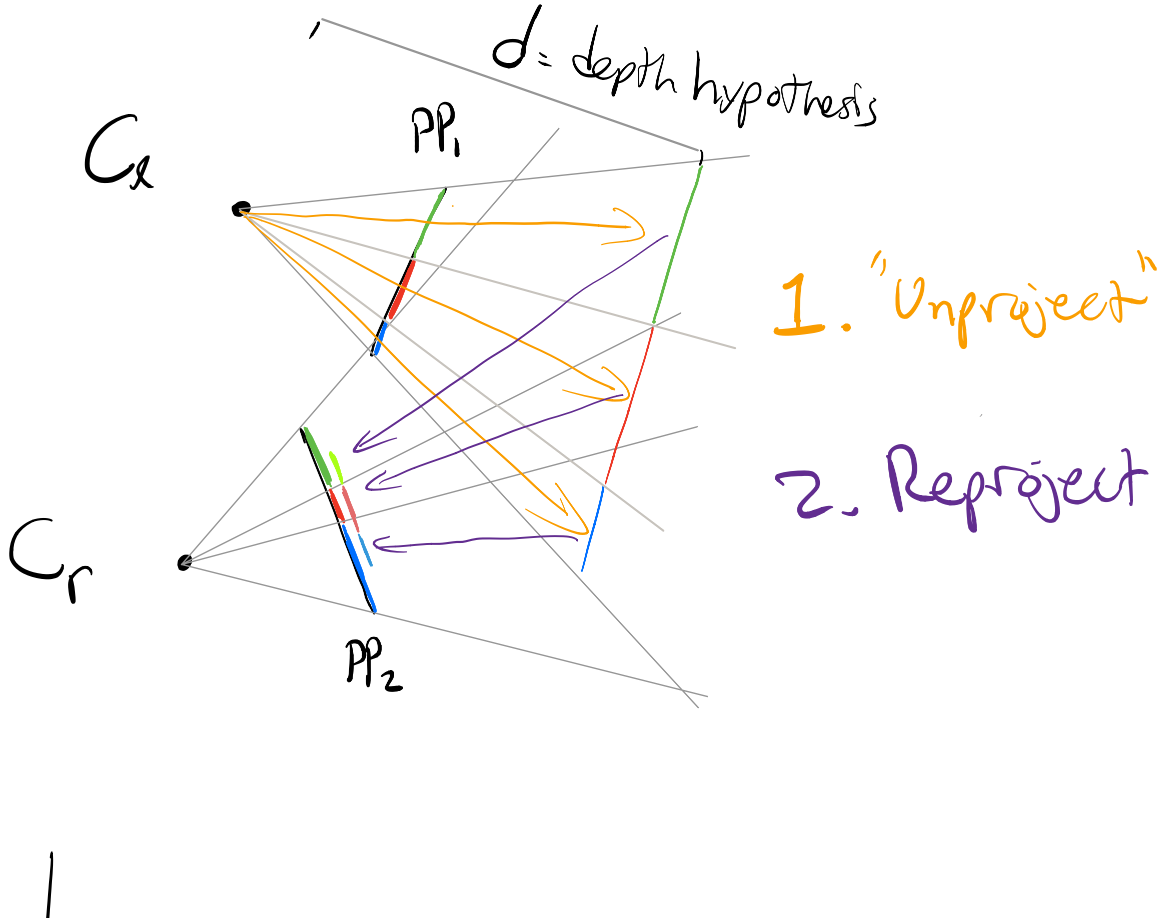
Goals (Today and Monday)

- Understand how lines are represented in projective space.
- Understand the duality of points and lines:
  - How to calculate the line through two points
  - How to check whether a point lies on a line
- Understand the derivation and significance of:
  - The Epipolar plane, epipolar lines, epipoles
  - The fundamental matrix
- Get a general sense for how camera parameters ([R|t], K) can be inferred from sets of feature matches.
- Know the definition of "structure from motion"

Example: A rectified stereo pair



Plane Sweep Stereo



$$\vec{P}_r = K_r \begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\end{bmatrix} \left[R_r \mid t_r\right] - t_\ell \; R_\ell^T \; d \; K_\ell^{-1} \; \vec{P}_\ell$$

img(r) — cam(r) — world (@ d) — ✗ 3D → 4D — cam(ℓ) — img(ℓ)

In Practice: Because the 3D object is Planar, we can move it arbitrarily with a <u>homography</u>.

1. Find the homography
   a. Unproject - Then reproject the 4 <u>corners</u> of C_ℓ
   b. Fit a homography from the 4 correspondences.

2. <u>Warp the left image onto the right image</u>

3. Compute NCC for the whole image and fill in the cost volume slice for depth d.

Let’s dig into that math hack of ours…

- (“whiteboard”)

Projective Geometry: Homogeneous Points

- whiteboard / lecture notes

- [x y 1] is equivalent all points [ax ay a]
- Viewed in 3D: All such points lie on a ray from [0 0 0] in the direction of [x y 1]
- Projective space considers points equal if they are equivalent under projection onto a plane.

Projective Geometry: Homogeneous Lines

- (see whiteboard/ lecture notes)

- Lines can also be represented in projective space (homogeneous coordinates).
- The line equation $ax + by + c = 0$ is represented using [a b c].
- Lines are invariant to scale as well: [ka kb kc] is the same as [a b c] for any k != 0.
- In the 3D view, a line represents all points on a plane through [0 0 0] whose **normal** is the vector [a b c].

Projective Geometry: Homogeneous Lines

- (see whiteboard/ lecture notes)

- What are the homogeneous (projective) coordinates for the following lines:
  - $y = -x$

Projective Geometry: Homogeneous Lines

- (see whiteboard/ lecture notes)

- What are the homogeneous (projective) coordinates for the following lines:
  - $y = -x$
  - $y = 2x + 4$

Projective Geometry: Homogeneous Lines

- (see whiteboard/ lecture notes)

- What are the homogeneous (projective) coordinates for the following lines:
  - $y = -x$
  - $y = 2x + 4$

- Can you write the same line with more than one 3-vector like you can with points?

Projective Geometry: Point-Line Duality

- (see whiteboard/ lecture notes)

- The **line** between two **points**, is the normal vector of the **plane** spanned by their **rays**.
- We can get a vector normal to a plane by taking the cross product of two vectors that span the plane.
- $l = p_1 \times p_2$

Projective Geometry: Point-Line Duality

- (see whiteboard/ lecture notes)

- To find the line through two points, take the cross-product of the points' homogeneous 3-vectors.
- How do you find the intersection point between two lines?

Projective Geometry: Points on Lines, Lines through Points

- (see whiteboard/ lecture notes)

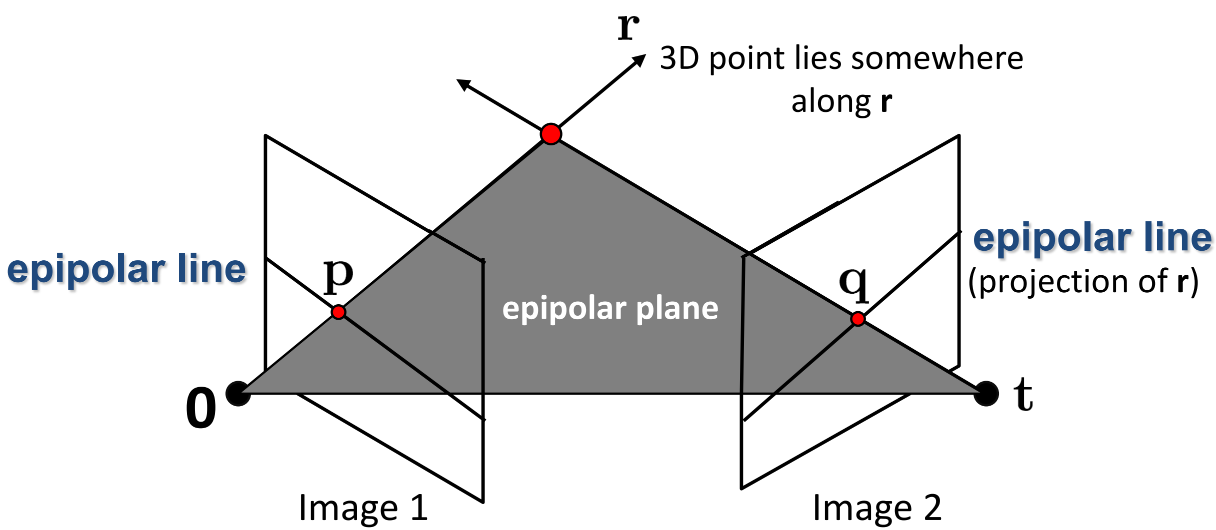
- A point [x y w] is on a line if:
  - $a(x/w) + b(y/w) + c = 0$ (multiply both sides by w)
  - $ax + by + cw = 0$
  - $[a\ b\ c] \cdot [x\ y\ w]^T = 0$
  - dot product!

Epipolar Geometry

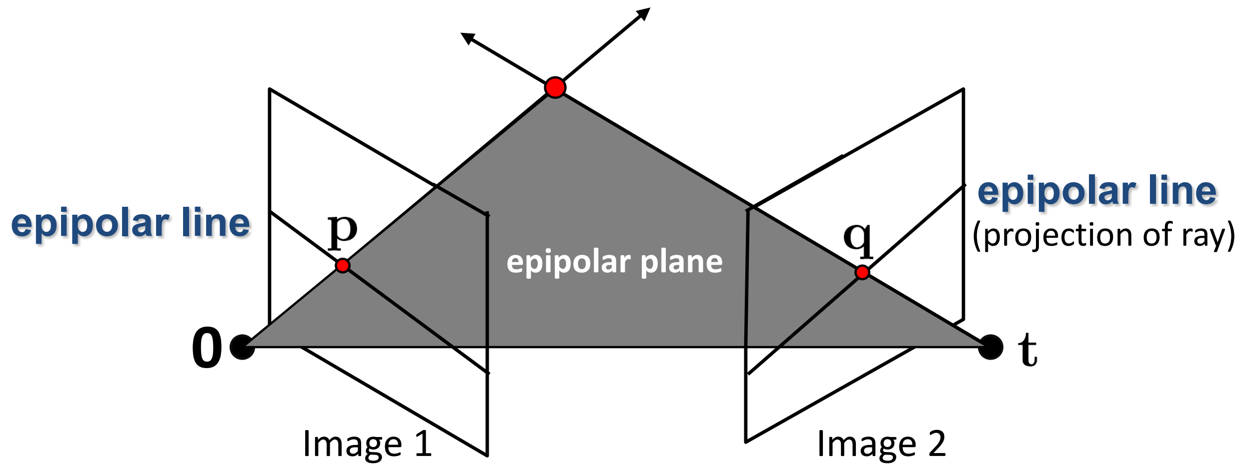
- Where could a point seen by one camera appear in a second camera?

Two-view geometry

- Where do epipolar lines come from?

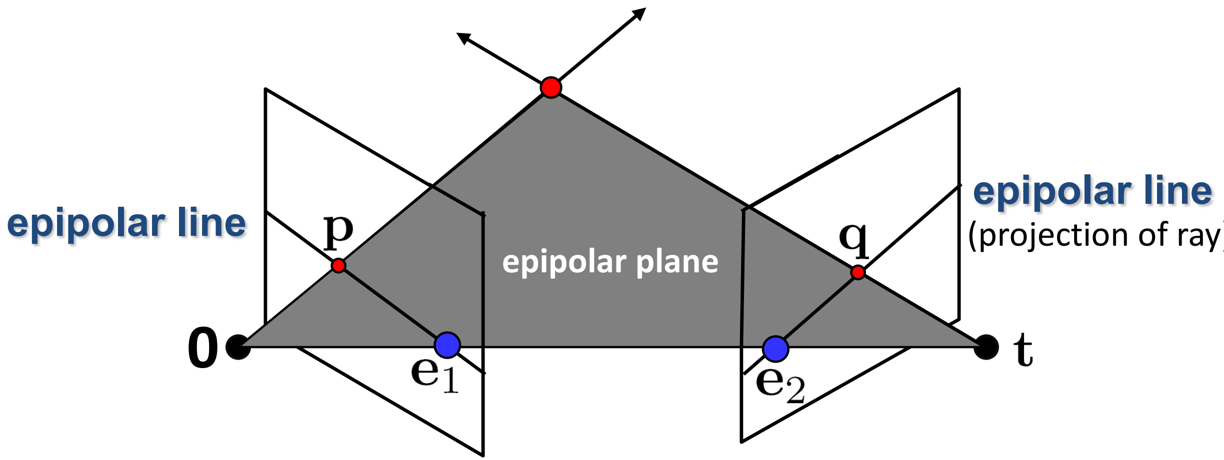


Fundamental matrix



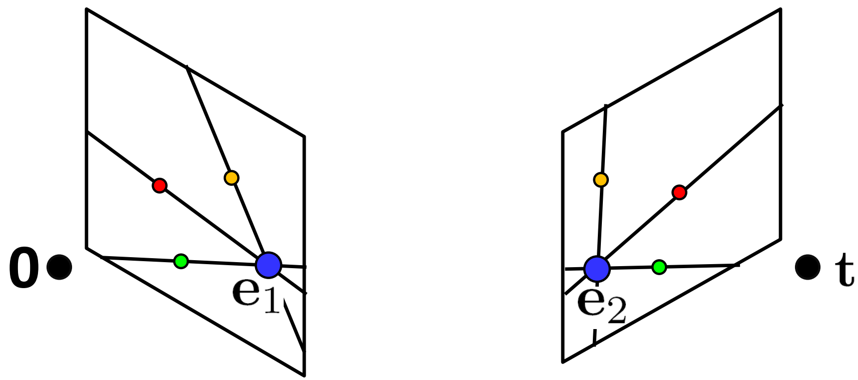
- This *epipolar geometry* of two views is described by a Very Special 3x3 matrix $\mathbf{F}$, called the *fundamental matrix*
- $\mathbf{F}$ maps (homogeneous) *points* in image 1 to *lines* in image 2!
- The epipolar line (in image 2) of point **p** is: $\mathbf{F}\mathbf{p}$
- *Epipolar constraint* on corresponding points: $\mathbf{q}^T\mathbf{F}\mathbf{p} = 0$

Fundamental matrix



- Two Special points: $\mathbf{e}_1$ and $\mathbf{e}_2$ (the *epipoles*): projection of one camera into the other

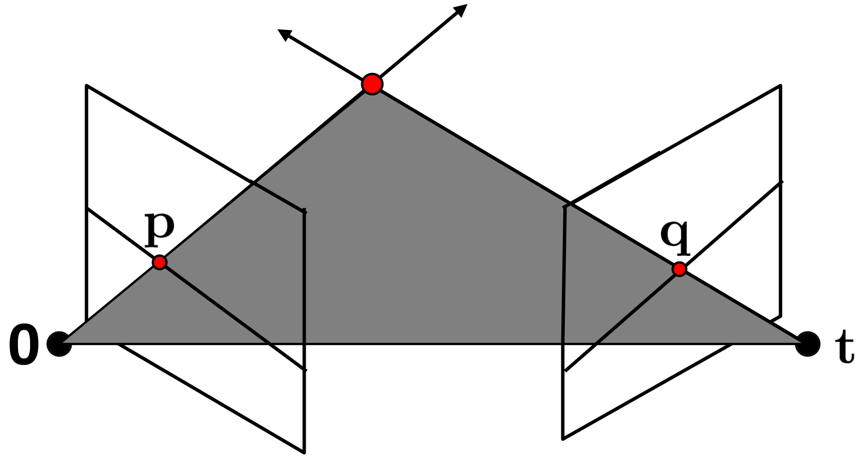
Fundamental matrix



- Two Special points: $\mathbf{e}_1$ and $\mathbf{e}_2$ (the *epipoles*): projection of one camera into the other
- All of the epipolar lines in an image pass through the epipole

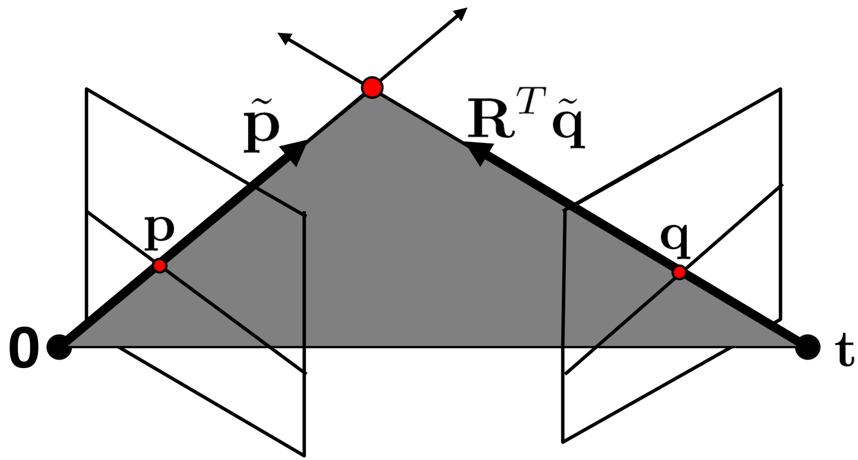
Epipoles

Fundamental matrix



- Why does **F** exist?
- Let's derive it…

Fundamental matrix – calibrated case



$\mathbf{K}_1$: intrinsics of camera 1

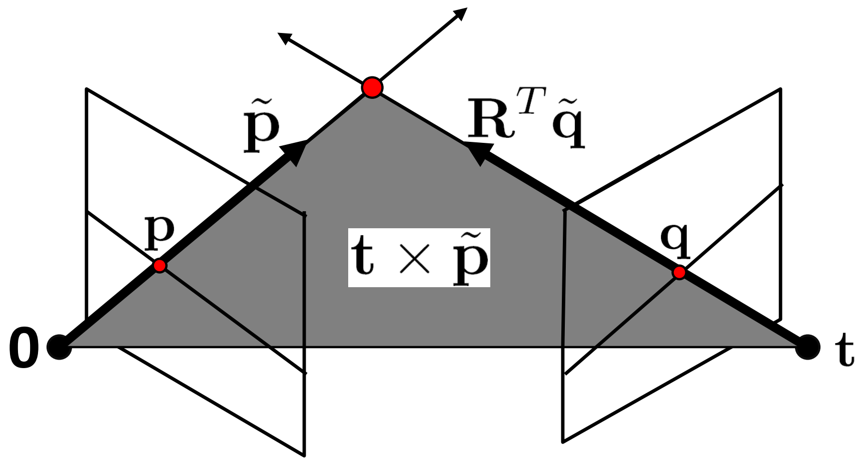
$\mathbf{K}_2$: intrinsics of camera 2

$\mathbf{R}$: rotation of image 2 w.r.t. camera 1

$\tilde{\mathbf{p}} = \mathbf{K}_1^{-1}\mathbf{p}$: ray through **p** in camera 1's (and world) coordinate system

$\tilde{\mathbf{q}} = \mathbf{K}_2^{-1}\mathbf{q}$: ray through **q** in camera 2's coordinate system

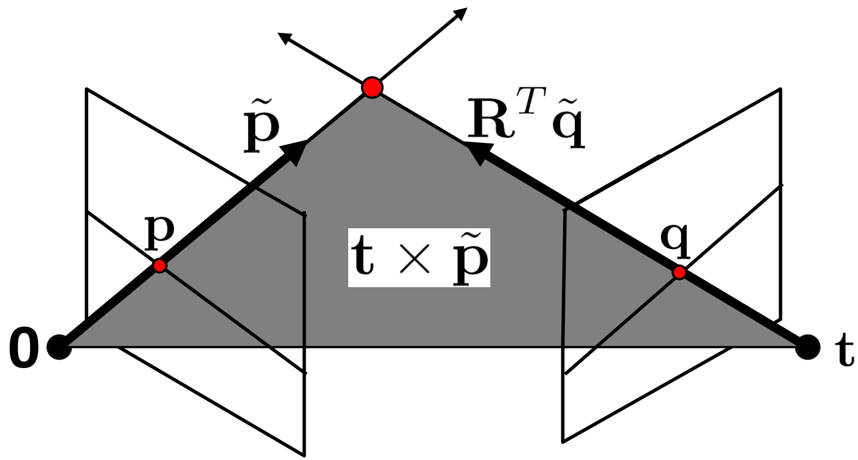
Fundamental matrix – calibrated case



- $\tilde{\mathbf{p}}$, $\mathbf{R}^T\tilde{\mathbf{q}}$, and $\mathbf{t}$ are coplanar
- epipolar plane can be represented as $\mathbf{t} \times \tilde{\mathbf{p}}$

$$(\mathbf{R}^T\tilde{\mathbf{q}})^T(\mathbf{t} \times \tilde{\mathbf{p}}) = 0$$

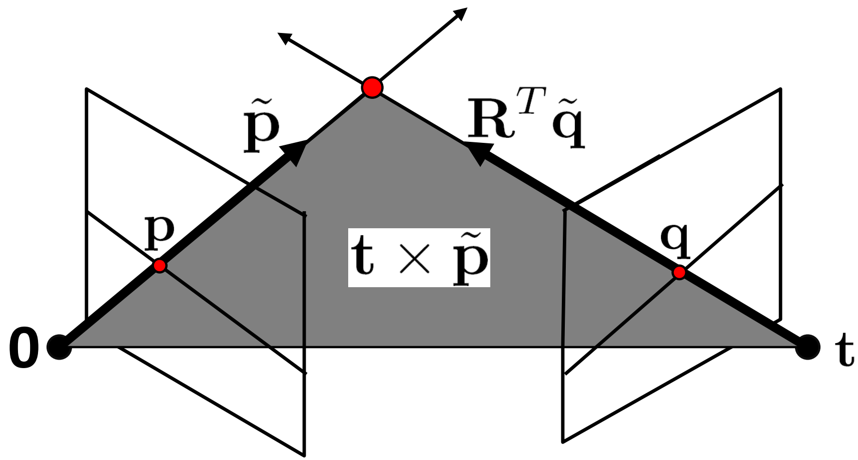
Fundamental matrix – calibrated case



$$(\mathbf{R}^T\tilde{\mathbf{q}})^T(\mathbf{t} \times \tilde{\mathbf{p}}) = 0$$

$$\tilde{\mathbf{q}}^T\mathbf{R}(\mathbf{t} \times \tilde{\mathbf{p}}) = 0$$

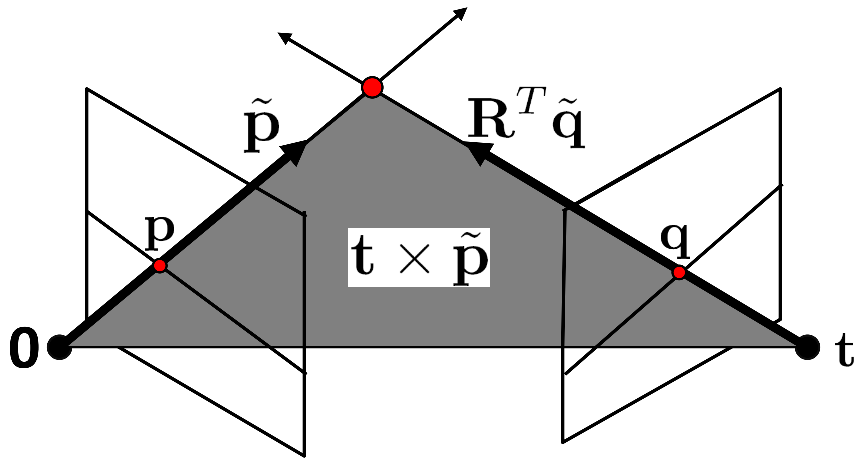
Fundamental matrix – calibrated case



- One more substitution:
  - Cross product with **t** can be represented as a 3x3 matrix

$$[\mathbf{t}]_\times = \begin{bmatrix} 0 & -t_z & t_y \\ t_z & 0 & -t_x \\ -t_y & t_x & 0 \end{bmatrix} \qquad \mathbf{t} \times \tilde{\mathbf{p}} = [\mathbf{t}]_\times \tilde{\mathbf{p}}$$

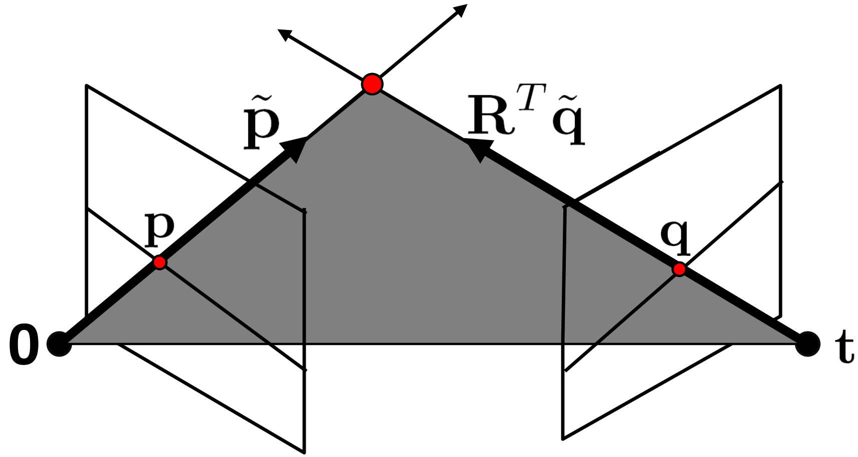
Fundamental matrix – calibrated case



$$\tilde{\mathbf{q}}^T \mathbf{R}(\mathbf{t} \times \tilde{\mathbf{p}}) = 0$$

$$\tilde{\mathbf{q}}^T \mathbf{R} [\mathbf{t}]_\times \tilde{\mathbf{p}} = 0$$

Fundamental matrix – calibrated case



$\tilde{\mathbf{p}} = \mathbf{K}_1^{-1}\mathbf{p}$: ray through **p** in camera 1's (and world) coordinate system

$\tilde{\mathbf{q}} = \mathbf{K}_2^{-1}\mathbf{q}$: ray through **q** in camera 2's coordinate system

$$\tilde{\mathbf{q}}^T \underbrace{\mathbf{R}\,[\mathbf{t}]_\times}_{\mathbf{E}} \tilde{\mathbf{p}} = 0$$

$$\tilde{\mathbf{q}}^T\mathbf{E}\tilde{\mathbf{p}} = 0$$

$\mathbf{E}$ ← the Essential matrix

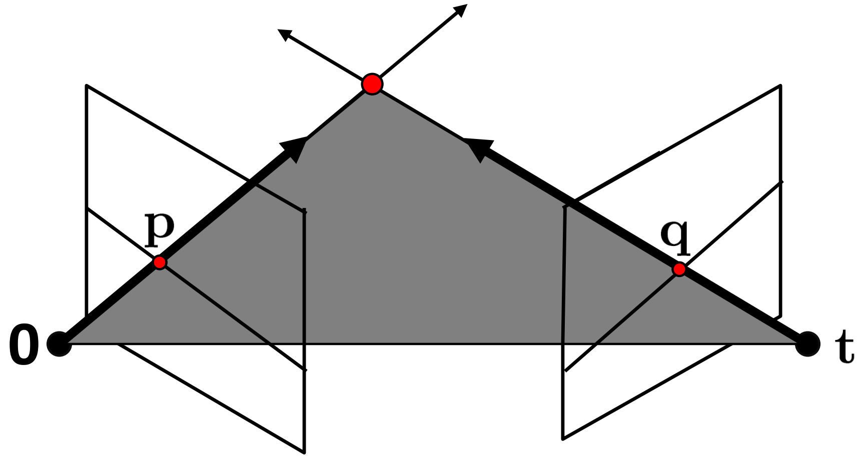
Cross-product as linear operator

Useful fact: Cross product with a vector **t** can be represented as multiplication with a (*skew-symmetric*) 3x3 matrix

$$[\mathbf{t}]_\times = \begin{bmatrix} 0 & -t_z & t_y \\ t_z & 0 & -t_x \\ -t_y & t_x & 0 \end{bmatrix}$$

$$\mathbf{t} \times \tilde{\mathbf{p}} = [\mathbf{t}]_\times \tilde{\mathbf{p}}$$

Fundamental matrix – uncalibrated case



$\mathbf{K}_1$: intrinsics of camera 1

$\mathbf{K}_2$: intrinsics of camera 2

$\mathbf{R}$: rotation of image 2 w.r.t. camera 1

$$\mathbf{q}^T \underbrace{\mathbf{K}_2^{-T} \mathbf{R} [\mathbf{t}]_\times \mathbf{K}_1^{-1}}_{\mathbf{F}} \mathbf{p} = 0$$

$\mathbf{F}$ ← the Fundamental matrix

Properties of the Fundamental Matrix

- $\mathbf{F}\mathbf{p}$ is the epipolar line associated with $\mathbf{p}$
- $\mathbf{F}^T\mathbf{q}$ is the epipolar line associated with $\mathbf{q}$
- $\mathbf{F}\mathbf{e}_1 = \mathbf{0}$ and $\mathbf{F}^T\mathbf{e}_2 = \mathbf{0}$
- $\mathbf{F}$ is rank 2
- How many parameters does **F** have?

Fundamental matrix song

https://www.youtube.com/watch?v=DgGV3l82NTk