Projective Geometry: Homogeneous Lines

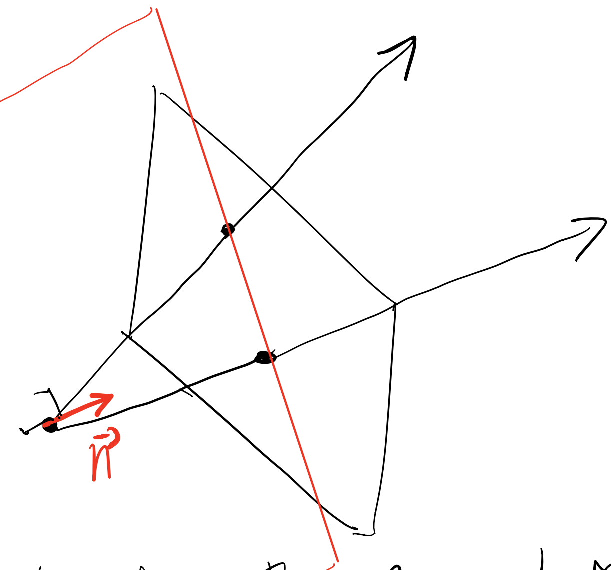
A point in $\mathbb{P}^2$ is a ray in 3D, projected onto a plane.

Can we represent lines in $\mathbb{P}^2$? Sure, why not.

A point is 0D object.
represented in $\mathbb{P}^2$ using a ray (1D)

A line is a 1D object.
represented in $\mathbb{P}^2$ using a plane (2D)

A line in $\mathbb{P}^2$ is the set of points that lie on a plane in $\mathbb{R}^3$ passing through the origin.



The line is represented using the normal vector of the plane.

$$\vec{n} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

In 2D, this projects to the line:

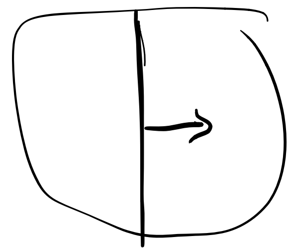
$$ax + by + c = 0$$

Examples:

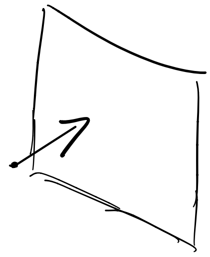
$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$1x + 0y + 0 = 0$$

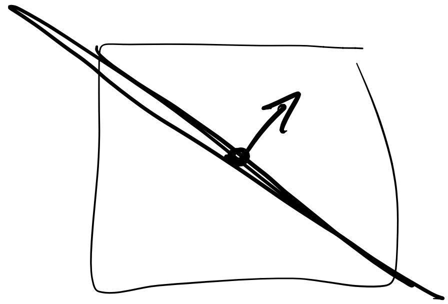
$$x = 0$$



$$\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$



$$y = -x$$

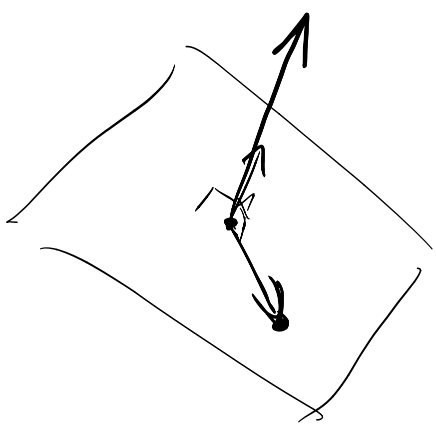


$$y = 2x + 4$$

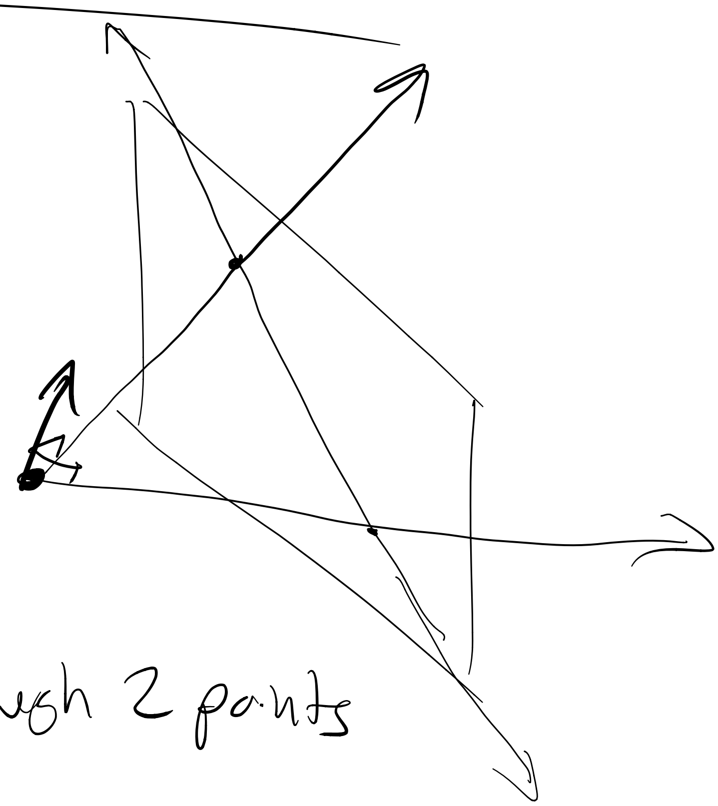
$$0 = \underset{a}{2x} \underset{b}{- y} + \underset{c}{4}$$

$$\begin{bmatrix} 2 \\ -1 \\ 4 \end{bmatrix}$$

$$\boxed{\begin{gathered} ax + by + c = 0 \\ kax + kby + kc = 0 \\ \text{if } k \neq 0 \end{gathered}}$$



Point-Line Duality



The line through 2 points is the plane spanned by their rays.

The plane normal vector is

$$\vec{l} = \vec{P}_1 \times \vec{P}_2$$

cross product!

Computing Cross Products

$$\begin{array}{ccccc} p_1 & & p_2 & & \\ \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} & & \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix} & = & \begin{bmatrix} y_1 z_2 - z_1 y_2 \\ z_1 x_2 - x_1 z_2 \\ x_1 y_2 - y_1 x_2 \end{bmatrix} \end{array}$$

yuck!

$$\begin{bmatrix} 0 & -z_1 & y_1 \\ z_1 & 0 & -x_1 \\ -y_1 & x_1 & 0 \end{bmatrix} \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix} = [\vec{p}_1]_x \cdot \vec{p}_2 =$$

If p is on l, how do I know?