Projective Geometry: Homogeneous Lines

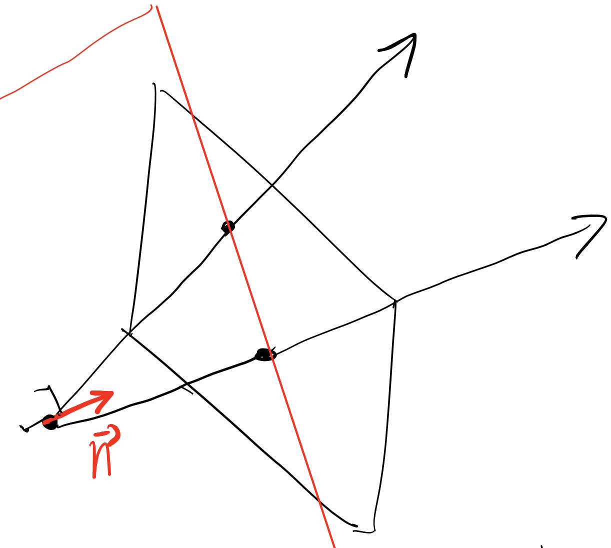
A point in $\mathbb{P}^2$ is a ray in 3D, projected onto a plane.

Can we represent lines in $\mathbb{P}^2$? Sure, why not.

A point is 0D object.
represented in $\mathbb{P}^2$ using a ray (1D)

A line is a 1D object.
represented in $\mathbb{P}^2$ using a plane (2D)

A line in $\mathbb{P}^2$ is the set of points that lie on a plane in $\mathbb{R}^3$ passing through the origin.



The line is represented using the normal vector of the plane.

$$\vec{n} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

In 2D, this projects to the line:

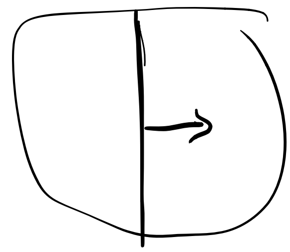
$$ax + by + c = 0$$

Examples:

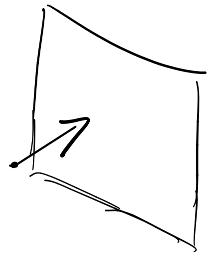
$$\begin{bmatrix}a\\b\\c\end{bmatrix} = \begin{bmatrix}1\\0\\0\end{bmatrix}$$

$$1x + 0y + 0 = 0$$

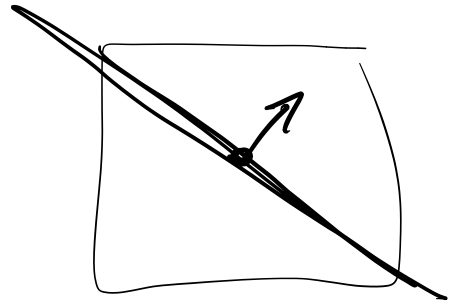
$$x = 0$$



$$\begin{bmatrix}1\\1\\0\end{bmatrix}$$



$$y = -x$$

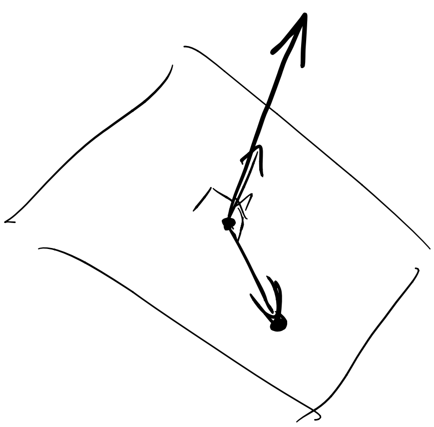


$$y = 2x + 4$$

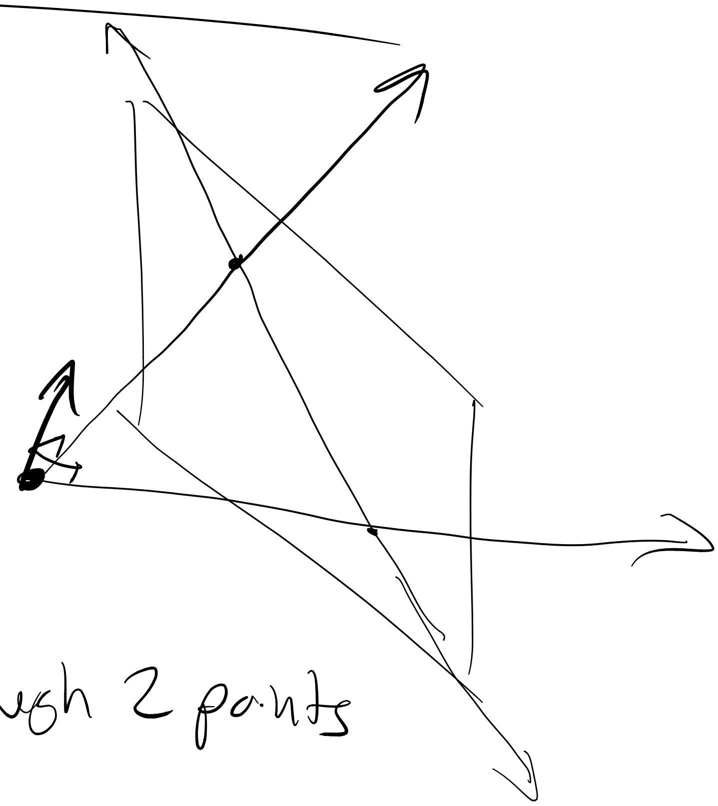
$$0 = \underset{a}{2x} \underset{b}{-y} \underset{c}{+4}$$

$$\begin{bmatrix}2\\-1\\4\end{bmatrix}$$

$$\boxed{\begin{array}{c} ax + by + c = 0 \\ kax + kby + kc = 0 \\ \text{if } k \neq 0 \end{array}}$$



Point-Line Duality



The line through 2 points

is the plane spanned by their rays.

The plane normal vector is

$$\vec{l} = \vec{P}_1 \times \vec{P}_2$$

cross product!

Computing Cross Products

$$\begin{array}{ccc} p_1 & p_2 & \\ \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} & \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix} & = \begin{bmatrix} y_1 z_2 - z_1 y_2 \\ z_1 x_2 - x_1 z_2 \\ x_1 y_2 - y_1 x_2 \end{bmatrix} \end{array}$$

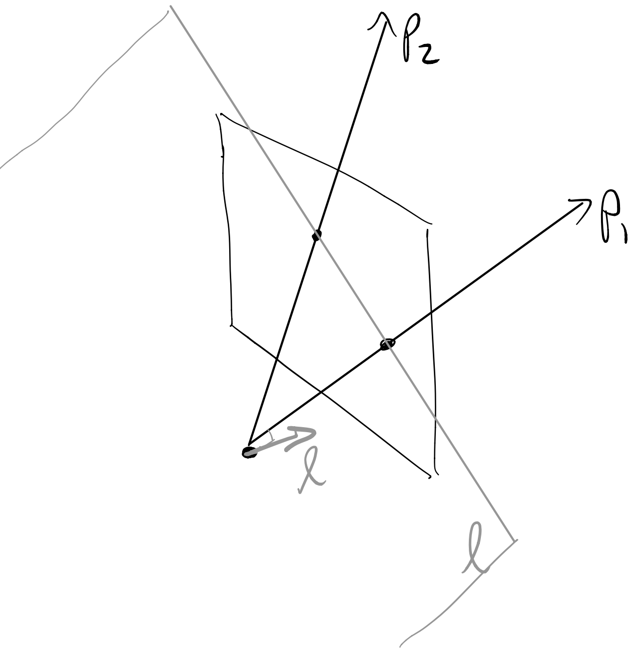
yuck!

$$\begin{bmatrix} 0 & -z_1 & y_1 \\ z_1 & 0 & -x_1 \\ -y_1 & x_1 & 0 \end{bmatrix} \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix} = [\vec{p}_1]_x \cdot \vec{p}_2 =$$

L24 - More Projective Geometry
The Fundamental Matrix

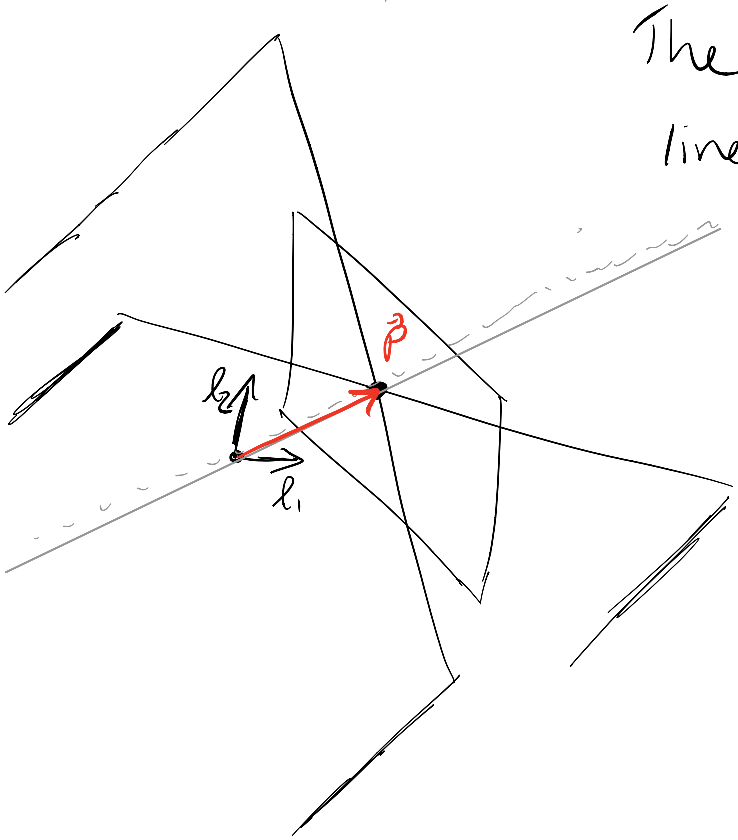
Announcements:

- Exam due 10pm tonight.
  - No slip days, no late submissions.
  - a few clarifications are on Piazza.
- P3 will be out tonight, due a week from tomorrow. (5/26)



The line through two points p_1, p_2 is a vector orthogonal to both point vectors:

$$l = p_1 \times p_2$$

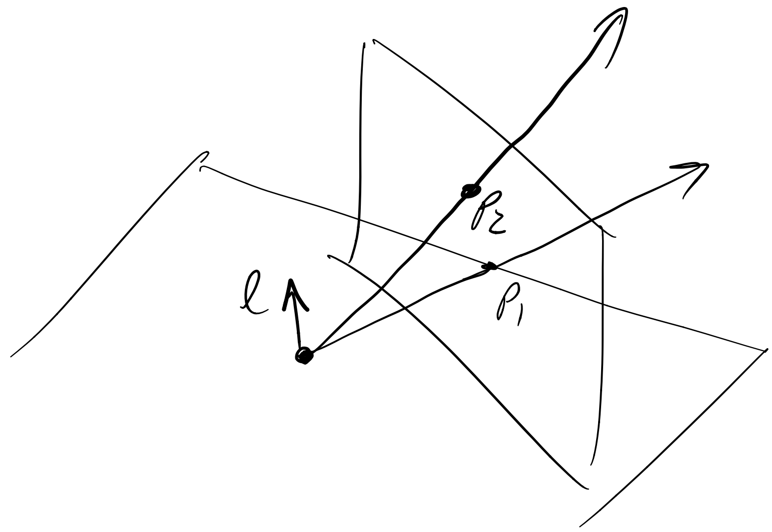


The point at the intersection of two lines l_1, l_2 is

$$p = l_1 \times l_2$$

How do I know if p is on ℓ?

Geometrically:



p_1 is on the plane

iff $p_1 \cdot \ell = 0$

ℓ goes through p iff

$\ell \cdot p = 0$

Algebraically:

$\ell = [a, b, c]$ represents $ax + by + c = 0$

$p = [x \ y \ w]$ represents $\left(\frac{x}{w}, \frac{y}{w}\right)$

$$a \frac{x}{w} + b \frac{y}{w} + c = 0$$

$$ax + by + cw = 0$$

$$= \begin{bmatrix} a & b & c \end{bmatrix} \begin{bmatrix} x \\ y \\ w \end{bmatrix} = 0$$

Epipolar Geometry

Let's add a second camera.

Assume:

$K_2 = I_{3\times3}$

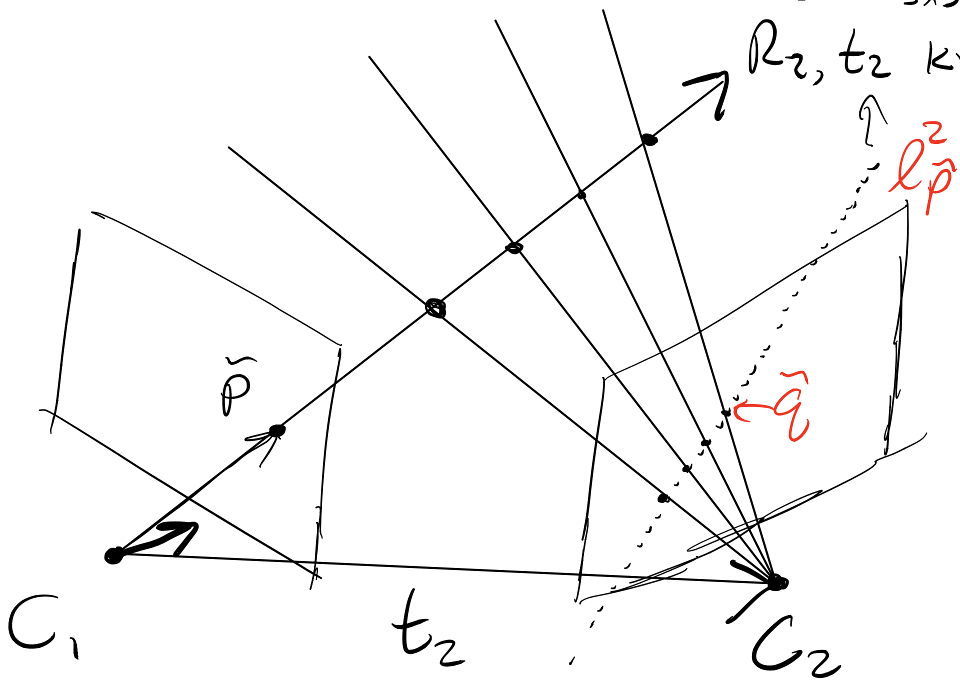
R_2, t_2 known

Assume:

$K_1 = I_{3\times3}$

$R_1 = I_{3\times3}$

$t_1 = \vec{0}$



Where does $\tilde{p}$ project in image 2?

- anywhere along the ray from C_1 to $\tilde{p}$

What's the equation (coordinates) of the line that projects to in C_2?

$$l^1_{\tilde{p}} = t_2 \times \tilde{p} \quad \leftarrow \text{in cam } 1!$$

$$l^2_{\tilde{p}} = R_2 (t_2 \times \tilde{p}) = \boxed{R_2 [t_2]_\times \tilde{p}}$$

We have: $l^2_{\tilde{p}} = R_2 [t_2]_\times \tilde{p}$

line in img2 where $\tilde{p}$'s ray projects

C_2's rotation

C_2's translation

point in img1

A point $\tilde{q}$ in img 2 that lies on $l^2_{\tilde{p}}$

satisfies: $\tilde{q} \cdot l^2_{\tilde{p}} = 0$

$$\text{or} \quad \underbrace{\tilde{q}^T}_{1\times 3} \boxed{\underbrace{R_2}_{3\times 3} \underbrace{[t_2]_\times}_{3\times 3}} \underbrace{\tilde{p}}_{3\times 1} = 0$$

Essential matrix

solve for

We assumed: $K_1 = K_2 = I_{3\times 3}$

Let $\hat{p} = K_1^{-1} p$

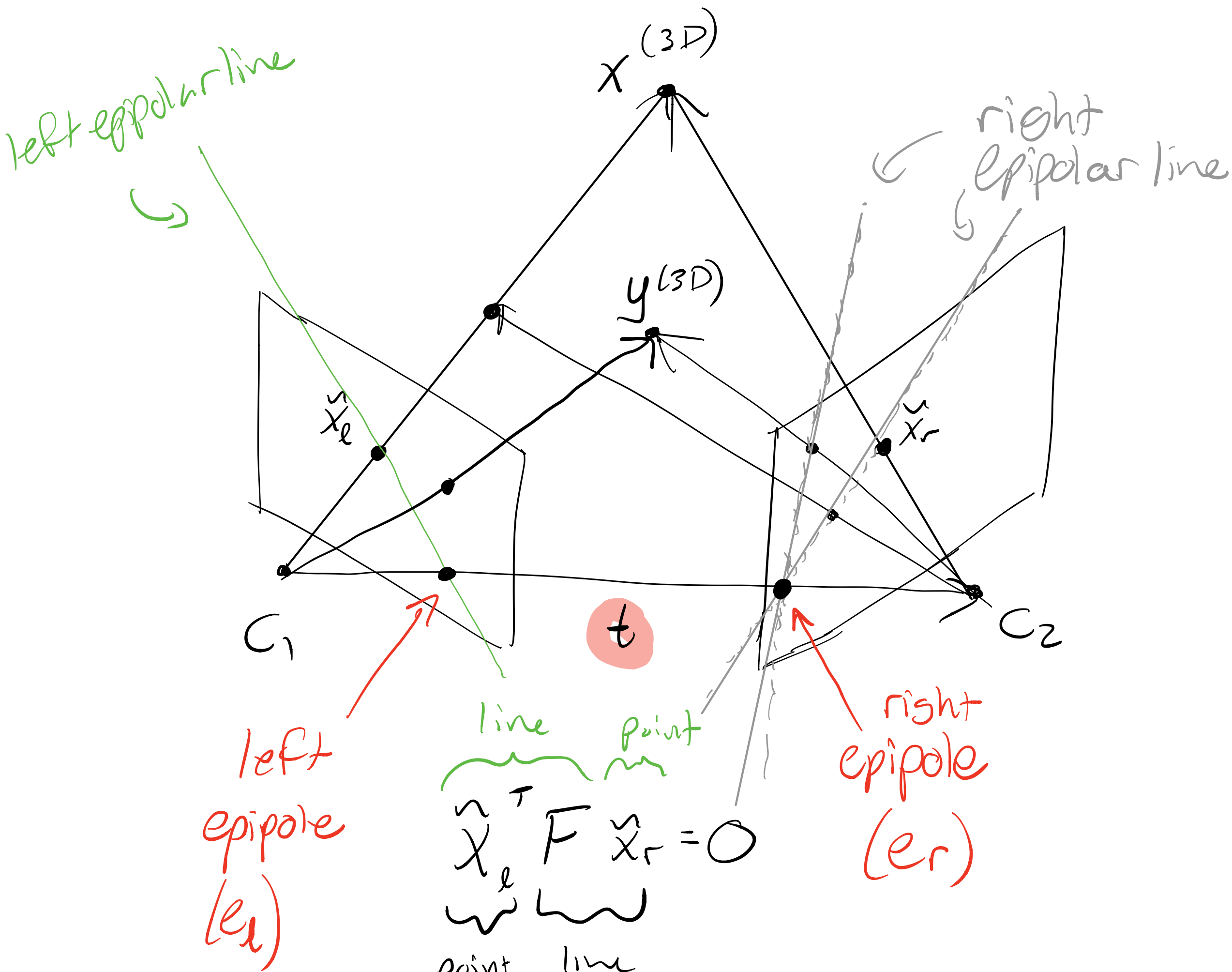
$\hat{q} = K_2^{-1} q$

$$q^T \underbrace{K_2^{-T} R_2 [t_2]_\times K_1^{-1}}_{F} p$$

Fundamental Matrix

$$q^T F p = 0$$

The epipolar constraint.



$$\overbrace{\underbrace{\hat{x}_\ell^T}_{\text{point}} F}^{\text{line}} \; \underbrace{\overbrace{\hat{x}_r}^{\text{point}}}_{} = 0$$

point line

Properties:

rank: # of lin. ind. cols.

$$\begin{bmatrix} | & | & | \\ c_1 & c_2 & \alpha c_1 + \beta c_2 \\ | & | & | \end{bmatrix}$$

- F has rank 2.

$$Fp \rightarrow 1D \text{ solution space}$$

$$q^T F p = 0$$

1D space of q's

- All epipolar lines go through the epipoles.

- e_ℓ spans the null space of F

$$F e_\ell = 0$$

- e_r spans the null space of F^T

Getting $K_\ell, K_r, R_\ell, R_r, t_\ell, t_r$
from F: hard, but possible