A point in $\mathbb{P}^2$ is a ray in 3D, projected onto a plane.

Can we represent lines in $\mathbb{P}^2$? Sure, why not.

A point is a 0D object represented in $\mathbb{P}^2$ using a ray (1D).

A line is a 1D object represented in $\mathbb{P}^2$ using a plane (2D).

A line in $\mathbb{P}^2$ is the set of points that lie on a plane in $\mathbb{R}^3$ passing through the origin.

The line is represented using the normal vector of the plane.

\[ \mathbf{n} = \begin{pmatrix} a \\ b \\ c \end{pmatrix} \]

In 2D, this projects to the line:

\[ ax + by + c = 0 \]
Examples:

\[
\begin{bmatrix}
1 \\
0 \\
0
\end{bmatrix}
\]

\[x + 0y + 0 = 0\]
\[x = 0\]

\[
\begin{bmatrix}
1 \\
0 \\
0
\end{bmatrix}
\]

\[y = -x\]

\[y = 2x + 4\]

\[
\begin{bmatrix}
2 \\
-1 \\
4
\end{bmatrix}
\]

\[ax + by + c = 0\]

\[k(ax + kby + kc) = 0\]

if \(k \neq 0\)
The line through 2 points is the plane spanned by their rays.

The plane normal vector is

\[ \hat{\mathbf{l}} = \hat{\mathbf{p}}_1 \times \hat{\mathbf{p}}_2 \]

cross product!

Computing Cross Products

\[
\begin{align*}
\left( \begin{array}{c}
x_1 \\
y_1 \\
z_1 \\
\end{array} \right) & \times \\
\left( \begin{array}{c}
x_2 \\
y_2 \\
z_2 \\
\end{array} \right) = \\
\left( \begin{array}{c}
y_1 z_2 - z_1 y_2 \\
z_1 x_2 - x_1 z_2 \\
x_1 y_2 - y_1 x_2 \\
\end{array} \right)
\end{align*}
\]

yuck!
$\begin{bmatrix} 0 & -z_1 & y_1 \\ z_1 & 0 & -x_1 \\ -y_1 & x_1 & 0 \end{bmatrix} \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix} = \begin{bmatrix} \hat{p}_1 \\ \hat{p}_2 \\ 1 \end{bmatrix}$

L24 - More Projective Geometry

The Fundamental Matrix

**Announcements:**

- Exam due 10pm tonight.
  - No slip days, no late submissions.
  - A few clarifications are on Piazza.
- P3 will be out tonight, due a week from tomorrow. (5/26)
The line through two points $p_1, p_2$ is a vector orthogonal to both point vectors:

$$l = p_1 \times p_2$$

The point at the intersection of two lines $l_1, l_2$ is

$$p = l_1 \times l_2$$
How do I know if \( p \) is on \( l \)?

**Geometrically:**

\( p \) is on the plane \( l \) iff \( p \cdot l = 0 \)

\( l \) goes through \( p \) iff \( l \cdot p = 0 \)

**Algebraically:**

\( l = [a, b, c] \) represents \( ax + by + c = 0 \)

\( p = [x, y, w] \) represents \( \left( \frac{x}{w}, \frac{y}{w} \right) \)

\[
a \frac{x}{w} + b \frac{y}{w} + c = 0
\]

\[
a x + b y + c w = 0
\]

\[= (a \ b \ c) \begin{pmatrix} x \\ y \\ w \end{pmatrix} = 0\]
Epipolar Geometry

Let's add a second camera. Assume:
\( K_2 = I_{3 \times 3} \)
\( R_2, t_2 \) known

Assume:
\( K_1 = I_{3 \times 3} \)
\( R_1, t_1 = 0 \)

Where does \( \tilde{p} \) project in image 2?
- Anywhere along the ray from \( C_1 \) to \( \tilde{p} \)

What's the equation (coordinates) of the line that projects \( \tilde{p} \) in image 2?

\[ l_\tilde{p}^1 = t_2 \times \tilde{p} \text{ in cam 1!} \]

\[ l_\tilde{p}^2 = R_2 \left( t_2 \times \tilde{p} \right) = R_2 \left( t_2 \times \tilde{p} \right) \]
We have: \( \ell_{\hat{p}}^2 = R_2[t_2] x \hat{p} \)

- Line in image 2 where \( \hat{p} \)'s ray projects
- \( C_2 \)'s rotation
- \( C_2 \)'s translation
- Point in image 1

A point \( \tilde{q} \) in image 2 that lies on \( \ell_{\hat{p}}^2 \) satisfies: \( \tilde{q} \cdot \ell_{\hat{p}}^2 = 0 \)

or \( \tilde{q}^T R_2(t_2) x \hat{p} = 0 \)

---

**Essential Matrix**

Solve for \( \tilde{q} \)

We assumed: \( k_1 = k_2 = I_{3 \times 3} \)
Let $\hat{p} = K_i^{-1} p$
$\hat{q} = K_2^{-1} q$

$Q^T K_2^{-T} R_2 [t^3_2] x K_i^{-1} p$

Fundamental Matrix

$q^T F p = 0$

The epipolar constraint.