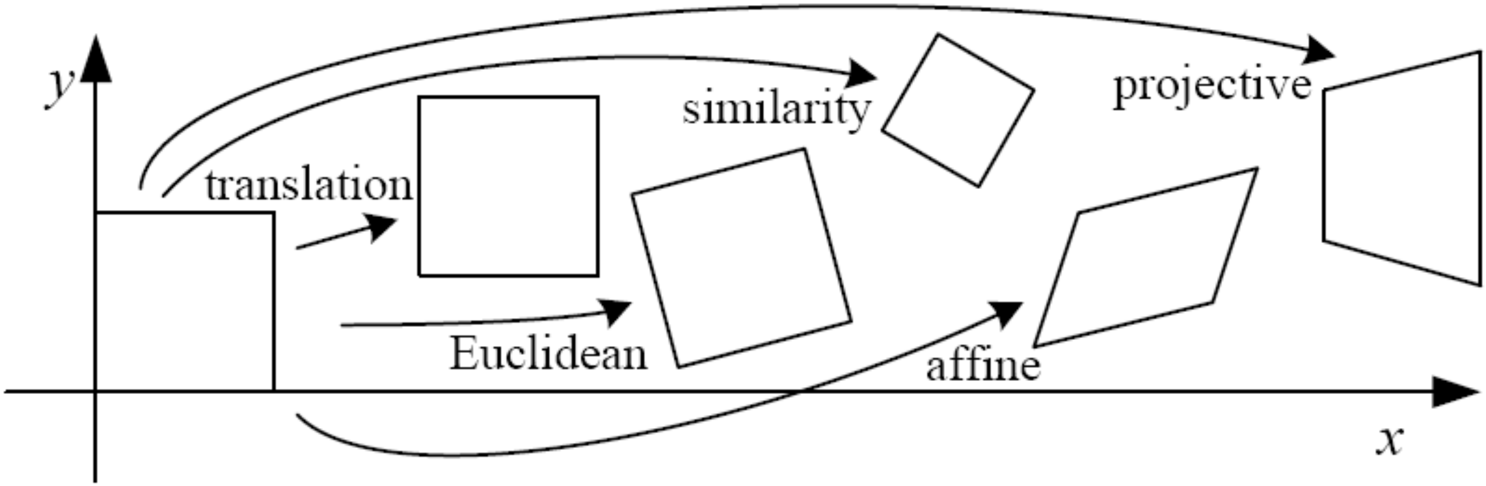
CSCI 497P/597P: Computer Vision



Lecture 15

Fitting Transformations:

Affine and Homography

Announcements

- Feedback survey
- You now have a total of 5 slip days, not 3. The counter does not reset.

Goals

- Know how to find a least-squares best-fit transformation for the following models:
  - translation
  - affine
  - homography

Transformations: Linear

$$\begin{bmatrix} x' \\ y' \\ w \end{bmatrix} = \begin{bmatrix} a & b & c \\ d & e & f \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ w \end{bmatrix}$$

- Properties:
  - Lines map to lines
  - Parallel lines remain parallel
  - Ratios of lengths along lines are preserved
  - Closed under composition
  - Origin maps to origin

linear

Transformations: Affine

$$\begin{bmatrix} x' \\ y' \\ w \end{bmatrix} = \begin{bmatrix} a & b & c \\ d & e & f \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ w \end{bmatrix}$$

- Properties:
  - Lines map to lines
  - Parallel lines remain parallel
  - Ratios of lengths along lines are preserved
  - Closed under composition
  - Origin **does not** necessarily map to origin

affine

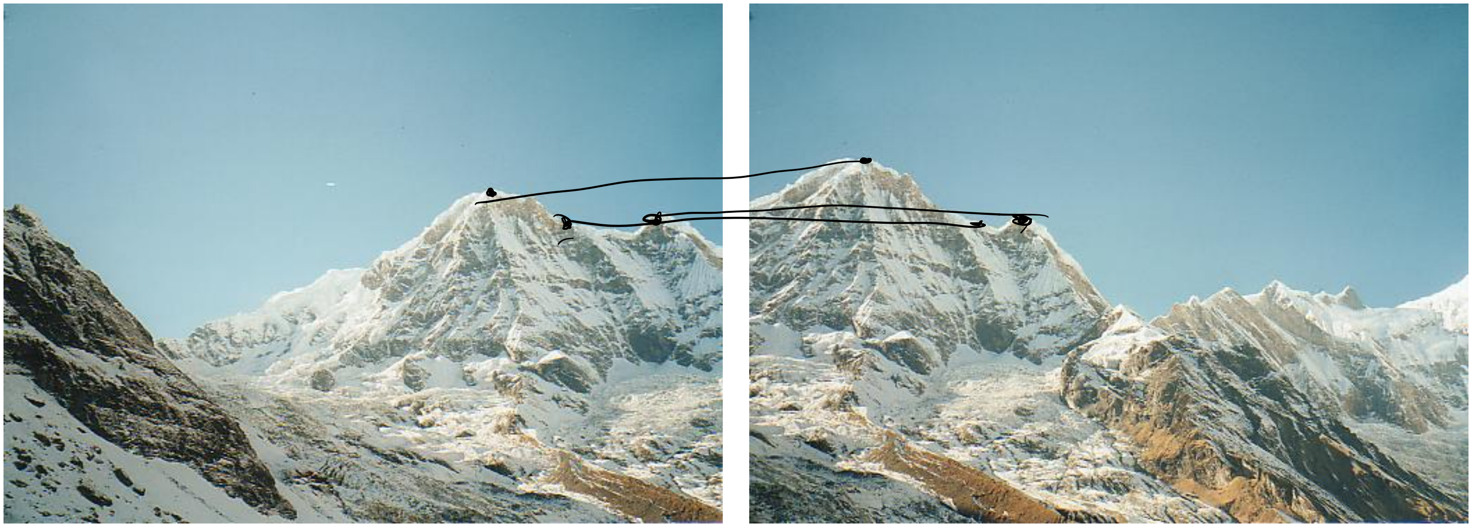
Transformations: Projective (Homography)

$$\begin{bmatrix} x' \\ y' \\ w' \end{bmatrix} = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ w \end{bmatrix}$$

- Properties:
  - Lines map to lines
  - Parallel lines **do not** remain parallel
  - Ratios of lengths along lines are **not** preserved.
  - Closed under composition
  - Origin **does not** necessarily map to origin

projective

Warping



We've found correspondence.
How do we find a transformation that "explains" the matches?

Problem Statement

(imperfect)

Given a set of feature matches, how do I **find** the **transformation** that relates the two images?

translation?
affine?
homography?

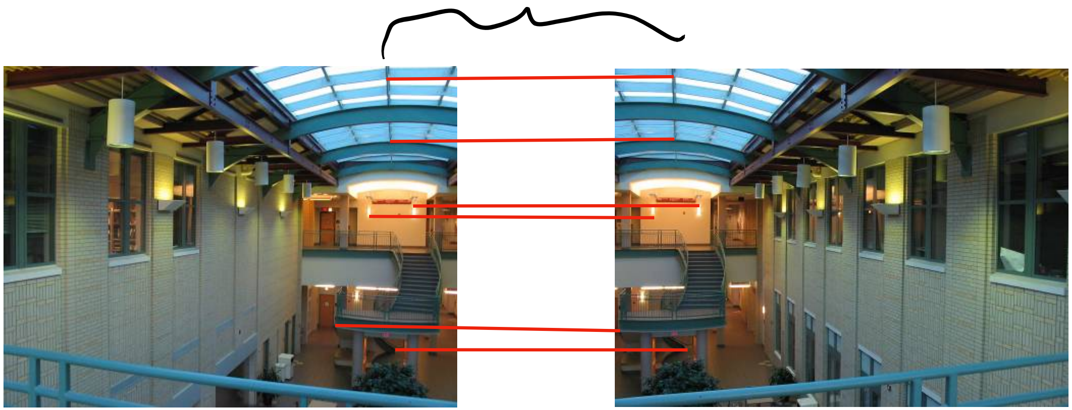


Image Alignment: Translation

$$\min_{t} \left\| \overset{2n\times 2}{A} \overset{2\times 1}{t} - \overset{2n\times 1}{b} \right\|$$

$$\min_{t} \left\| \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \\ 0 & 1 \\ \vdots & \vdots \end{bmatrix} \begin{bmatrix} t_x \\ t_y \end{bmatrix} - \begin{bmatrix} x_1' - x_1 \\ y_1' - y_1 \\ x_2' - x_2 \\ y_2' - y_2 \\ \vdots \end{bmatrix} \right\|$$

$t_x - (x_1' - x_1)$

$t_y - (y_1' - y_1)$

Image Alignment: Affine

point in img2 → $\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} a & b & c \\ d & e & f \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$ ← point in img 1

$$r_{x_i}(t) \quad (ax_i + by_i + c) - x_i'$$

$$(dx_i + ey_i + f) - y_i'$$

$$\min_t \|A\overset{\text{unk}}{t} - b\|^2$$

$$\left\| \overset{2n \times 6}{\begin{bmatrix} x_1 & y_1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & x_1 & y_1 & 1 \\ x_2 & y_2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & x_2 & y_2 & 1 \\ & & \vdots & & & \end{bmatrix}} \overset{6 \times 1}{\begin{bmatrix} a \\ b \\ c \\ d \\ e \\ f \end{bmatrix}} - \overset{2n \times 1}{\begin{bmatrix} x_1' \\ y_1' \\ x_2' \\ y_2' \\ \vdots \end{bmatrix}} \right\|^2$$

A = . . .

b = . .

t = np.linalg.lstsq(A, b)

Image Alignment: Homography

$$\begin{bmatrix} x_h \\ y_h \\ w_h \end{bmatrix} = \begin{bmatrix} h_{00} & h_{01} & h_{02} \\ h_{10} & h_{11} & h_{12} \\ h_{20} & h_{21} & h_{22} \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

True residual:

$$\boxed{\frac{x_h}{w_h} - x'}$$

not linear! :(

$$h_{00}x + h_{01}y + h_{02} - x'(h_{20}x + h_{21}y + h_{22})$$

$$h_{10}x + h_{11}y + h_{12} - y'(h_{20}x + h_{21}y + 1)$$

Approximate residual:

$$\boxed{x_h - x'(w_h)}$$

:)

linear in h_{ij}

$$\min_h \|Ah - b\|^2$$

$$
\begin{bmatrix}
x_1 & y_1 & 1 & 0 & 0 & 0 & -(x_1 x_1') & -(x_1' y_1) & -x_1' \\
0 & 0 & 0 & x_1 & y_1 & 1 & -x_1 y_1' & -y_1 y_1' & -y_1' \\
x_2 & y_2 & 1 & 0 & 0 & 0 & -x_2 x_2' & -x_2' y_2 & -x_2' \\
 & & & & \vdots & & & &
\end{bmatrix}
\begin{bmatrix}
h_{00} \\ h_{01} \\ h_{02} \\ h_{10} \\ h_{11} \\ h_{12} \\ h_{20} \\ h_{21} \\ h_{22}
\end{bmatrix}
-
\begin{bmatrix}
0 \\ 0 \\ 0 \\ 0 \\ 0 \\ \vdots
\end{bmatrix}
$$

Solving $\min_h \|Ah - 0\|^2$

2n x 1 vector of zeros

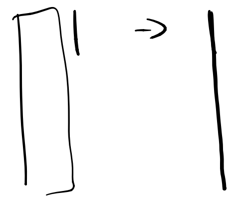
$$\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ \vdots \end{bmatrix}$$

= np.linalg.lstsq(A, np.zeros((2n,)))

Approach: $\|h\| = 1$ as an added constraint subject to

$$\min_h \|Ah - 0\|^2 \quad \text{S.T} \quad \|h\| = 1$$

$$\min_h \underbrace{\|Ah\|^2}$$



$$x^T x$$

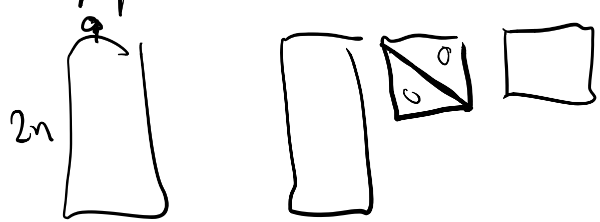
$$-1 \rightarrow \|x\|^2$$

$$(Ah)^T(Ah) = (h^T \underbrace{A^T A} h)$$

$=$

Singular Value Decomposition (SVD)

decompose $A = U \Sigma V^T$



U, V : unitary, orthogonal:

$u_i^T u_j = 0 \quad i \neq j$

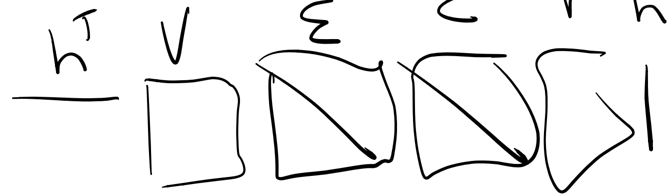
$u_i^T u_i = 1$

$$\underset{2n\times 9}{} \quad \underset{9\times 9}{} \quad \underset{9\times 9}{}$$

$$h^T V \Sigma^T \overbrace{(U^T U)}^{I} \Sigma V^T h$$

$$\left(h^T V \Sigma^T \underbrace{\Sigma V^T h}\right)$$

$\Sigma^T = \Sigma$ because Σ is diagonal



$$\left\|\Sigma V^T h\right\|^2$$

diag $\quad \|v_i\| = 1 \quad \|h\| = 1$

$$\begin{bmatrix} \sigma_1 & & & \\ & \sigma_2 & & \\ & & \ddots & \\ & & & \sigma_9 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_9 \end{bmatrix} \begin{bmatrix} h \end{bmatrix}$$

h that minimizes squared residuals is the vector v_i corresponding to the smallest σ_i (entry of Σ)

TL;DM

1. Compute SVD of A
2. Find index i of smalles σ
3. Take the ith column of V as h.
4. Reshape to 3×3 and divide by h_{22}.