Lecture 14
Homogeneous Points Intuition
Fitting Transformations
Announcements
Goals

- Know the definition of a projective (homography) transformation, and gain some geometric intuition for what it represents in 2D.
  - Understand the meaning of homogeneous points at infinity.
  - Know why a homography has 8 degrees of freedom, not 9 [hw2]

- Know how to find a least-squares best-fit transformation for the given models: translation, affine, homography
Transformations: Linear

Properties:
- Lines map to lines
- Parallel lines remain parallel
- Ratios of lengths along lines are preserved
- Closed under composition
- Origin maps to origin

\[ \begin{bmatrix} x' \\ y' \\ w \end{bmatrix} = \begin{bmatrix} a & b & c \\ d & e & f \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ w \end{bmatrix} \]
Transformations: Affine

\[
\begin{bmatrix}
  x' \\
  y' \\
  w
\end{bmatrix}
= 
\begin{bmatrix}
  a & b & c \\
  d & e & f \\
  0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
  x \\
  y \\
  w
\end{bmatrix}
\]

- **Properties:**
  - Lines map to lines
  - Parallel lines remain parallel
  - Ratios of lengths along lines are preserved
  - Closed under composition
  - Origin does not necessarily map to origin
Transformations: Projective (Homography)

- Properties:
  - Lines map to lines
  - Parallel lines **do not** remain parallel
  - Ratios of lengths along lines are **not** preserved.
  - Closed under composition
  - Origin **does not** necessarily map to origin

\[
\begin{bmatrix}
x' \\
y' \\
w'
\end{bmatrix} =
\begin{bmatrix}
a & b & c \\
d & e & f \\
g & h & 1
\end{bmatrix}
\begin{bmatrix}
x \\
y \\
w
\end{bmatrix}
\]
Homogeneous Coordinates: Intuition for our math hack

A 3-vector belongs to a **family** of 3-vectors representing the same 2D point.

1D example, for intuition: A 2-vector belongs to a **family** of 2-vectors representing the same 1D point.
In 2D: (Pretend-3D)

Affine:
\[
\begin{bmatrix}
a & b & c \\
d & e & f \\
1 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
x \\
y \\
1
\end{bmatrix}
\]

Homography:
\[
\begin{bmatrix}
a & b & c \\
d & e & f \\
g & h & 1
\end{bmatrix}
\begin{bmatrix}
x \\
y \\
1
\end{bmatrix}
\]
What are we looking at?
Transformations: A Hierarchy

<table>
<thead>
<tr>
<th>Name</th>
<th>Matrix</th>
<th># D.O.F.</th>
<th>Preserves:</th>
<th>Icon</th>
</tr>
</thead>
<tbody>
<tr>
<td>translation</td>
<td>$[I \mid t]_{2\times3}$</td>
<td>2</td>
<td>orientation + ⋯</td>
<td></td>
</tr>
<tr>
<td>rigid (Euclidean)</td>
<td>$[R \mid t]_{2\times3}$</td>
<td>3</td>
<td>lengths + ⋯</td>
<td>⬤</td>
</tr>
<tr>
<td>similarity</td>
<td>$[sR \mid t]_{2\times3}$</td>
<td>4</td>
<td>angles + ⋯</td>
<td>⬤</td>
</tr>
<tr>
<td>affine</td>
<td>$[A]_{2\times3}$</td>
<td>6</td>
<td>parallelism + ⋯</td>
<td>⬤</td>
</tr>
<tr>
<td>projective</td>
<td>$[\tilde{H}]_{3\times3}$</td>
<td>8</td>
<td>straight lines</td>
<td>⬤</td>
</tr>
</tbody>
</table>
Homographies for image alignment
Why the 1 in the bottom right?

Prove on HW2: For any 3x3 matrix $H$, there exists another 3x3 matrix $H'$ which:

- Has a 1 in the bottom-right corner
- Has the same effect on homogeneous points.
Why the 1 in the bottom right?

Prove on HW2: For any 3x3 matrix $H$, there exists another 3x3 matrix $H'$ which:

- Has a 1 in the bottom-right corner
- Has the same effect on homogeneous points.

Fixing the 1 is merely a convention.
Another convention: values in $H$ as a vector have magnitude 1.
Why the 1 in the bottom right?

Prove on HW2: For any 3x3 matrix $H$, there exists another 3x3 matrix $H'$ which:

- Has a 1 in the bottom-right corner
- Has the same effect on homogeneous points.

Consequently:
- homographies have 8 degrees of freedom (DOF).
- affine transformations have 6 DOF
- translations have 2 DOF
About those parallel lines...

The homography $H$ transforms points from the left image to the right image.

What is $H$? 

$$
\begin{pmatrix}
X \\
Y \\
1
\end{pmatrix}
\overset{H}{\rightarrow}
\begin{pmatrix}
0 \\
0 \\
-1
\end{pmatrix}
$$
point at unit [0]
Next up:

(imperfect)

Given a set of feature matches, how do I find the transformation that relates the two images?
Image Alignment: Translation

\[ t_x = x'_1 - x, \]
\[ t_y = y'_1 - y, \]

\[
\begin{bmatrix}
1 & 0 & t_x \\
0 & 1 & t_y \\
0 & 0 & 1
\end{bmatrix}
\]

\[ (x, y, x'_1, y'_1) \]
Multiple Matches?

\[
\begin{align*}
    t_x &= \frac{1}{n} \sum_{i=0}^{n} (x_i' - x_i) \\
    t_y &= \frac{1}{n} \sum_{i=0}^{n} (y_i' - y_i)
\end{align*}
\]

Multiple Matches: Linear Algebra Edition!

Goal: Minimize difference between observed landing points and model's prediction:

\[
\sum_{i=0}^{n} \left( \begin{bmatrix} x_i' \\ y_i' \end{bmatrix} - \begin{bmatrix} x_i + t_x \\ y_i + t_y \end{bmatrix} \right)^2
\]
$$\min_x \| A x - b \|^2$$

$$A x = b$$

For Translation:

$$\| r_i (t_x) \|^2 = (x_i - (x_i + t_x))^2$$

$$\| r_i (t_y) \|^2 = (y_i - (y_i + t_y))^2$$
\[
\min_{x} \mathbf{A}x - b
\]

\[
\begin{bmatrix}
1 \\
0 \\
1 \\
0 \\
... \\
0
\end{bmatrix}
\]

\[
\begin{bmatrix}
t_x \\
t_y
\end{bmatrix}
\]

\[
\begin{bmatrix}
x_1' - x_1 \\
y_1' - y_1 \\
x_2' - x_2 \\
y_2' - y_2 \\
... \\
0
\end{bmatrix}
\]

\[
t_x - (x_1' - x_1) \\
t_y - (y_1' - y_1)
\]
Solving \( \min_x \|Ax - b\|^2 \)

Math class: normal equations

\[(A^TA)^{-1}(A^T A)x = A^T b\]

\[x = (A^TA)^{-1}A^T b\]

On a computer:

`np.linalg.lstsq(A, b)`

Use this!