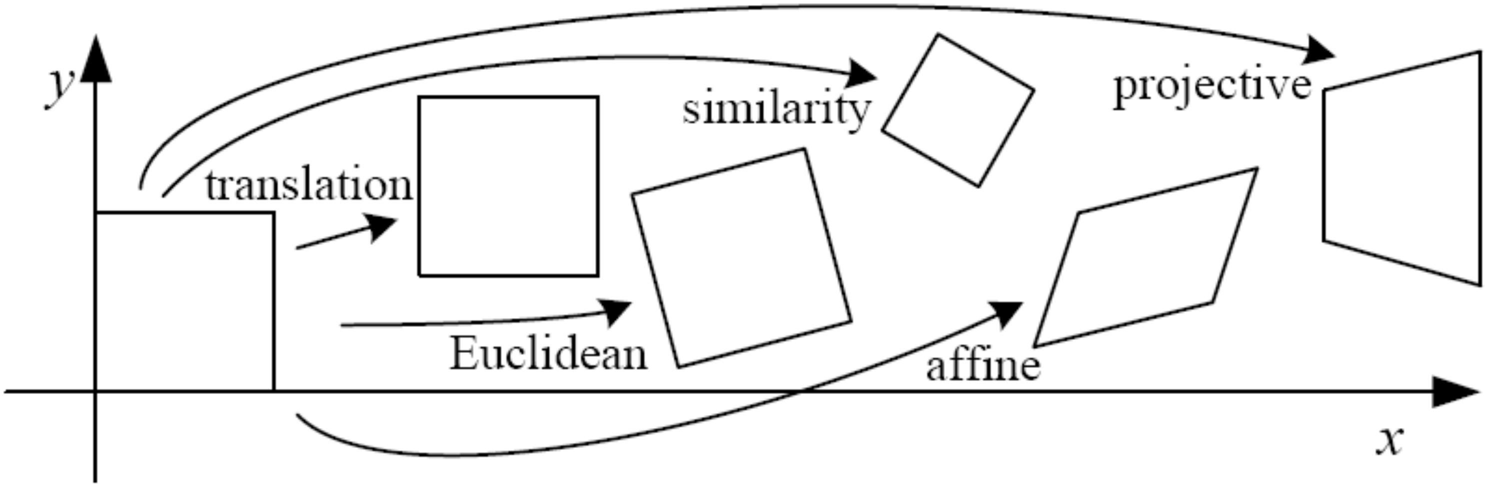
CSCI 497P/597P: Computer Vision



Lecture 14

Homogeneous Points Intuition

Fitting Transformations

Announcements

Goals

- Know the definition of a projective (homography) transformation, and gain some geometric intuition for what it represents in 2D.
  - Understand the meaning of homogeneous points at infinity.
  - Know why a homography has 8 degrees of freedom, not 9 [hw2]
- Know how to find a least-squares best-fit transformation for the given models: translation, affine, homography

Transformations: Linear

$$\begin{bmatrix} x' \\ y' \\ w \end{bmatrix} = \begin{bmatrix} a & b & c \\ d & e & f \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ w \end{bmatrix}$$

- Properties:
  - Lines map to lines
  - Parallel lines remain parallel
  - Ratios of lengths along lines are preserved
  - Closed under composition
  - Origin maps to origin

linear

Transformations: Affine

$$\begin{bmatrix} x' \\ y' \\ w \end{bmatrix} = \begin{bmatrix} a & b & c \\ d & e & f \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ w \end{bmatrix}$$

- Properties:
  - Lines map to lines
  - Parallel lines remain parallel
  - Ratios of lengths along lines are preserved
  - Closed under composition
  - Origin **does not** necessarily map to origin

affine

Transformations: Projective (Homography)

$\frac{x}{w}$ $\frac{y}{w}$

$$\begin{bmatrix} x' \\ y' \\ w' \end{bmatrix} = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ w \end{bmatrix}$$

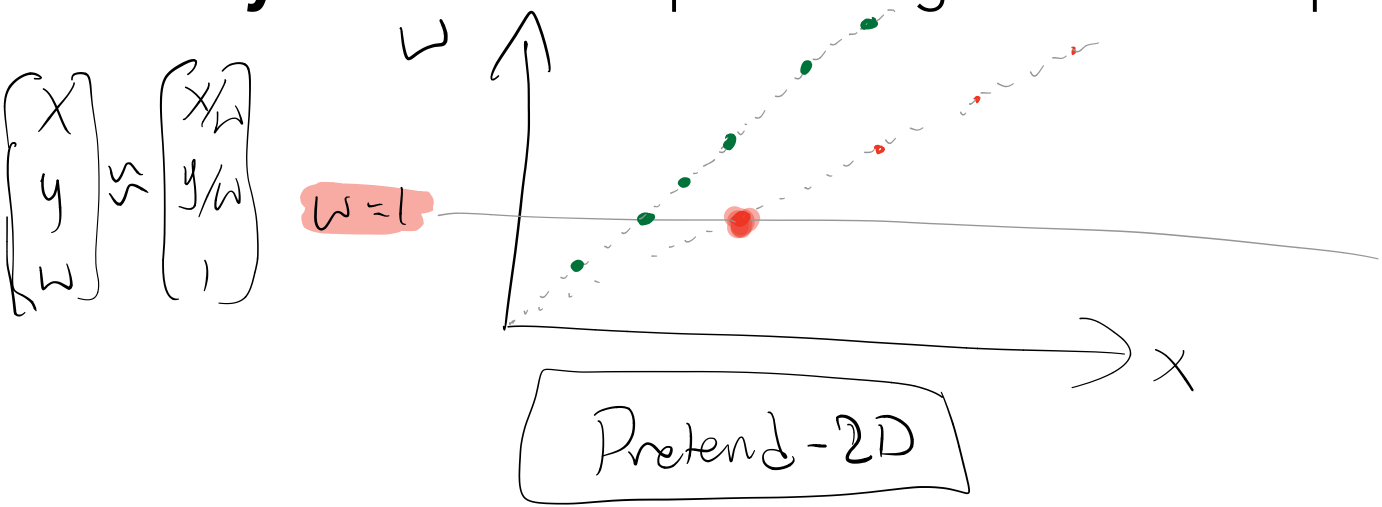
- Properties:
  - Lines map to lines
  - Parallel lines **do not** remain parallel
  - Ratios of lengths along lines are **not** preserved.
  - Closed under composition
  - Origin **does not** necessarily map to origin

projective

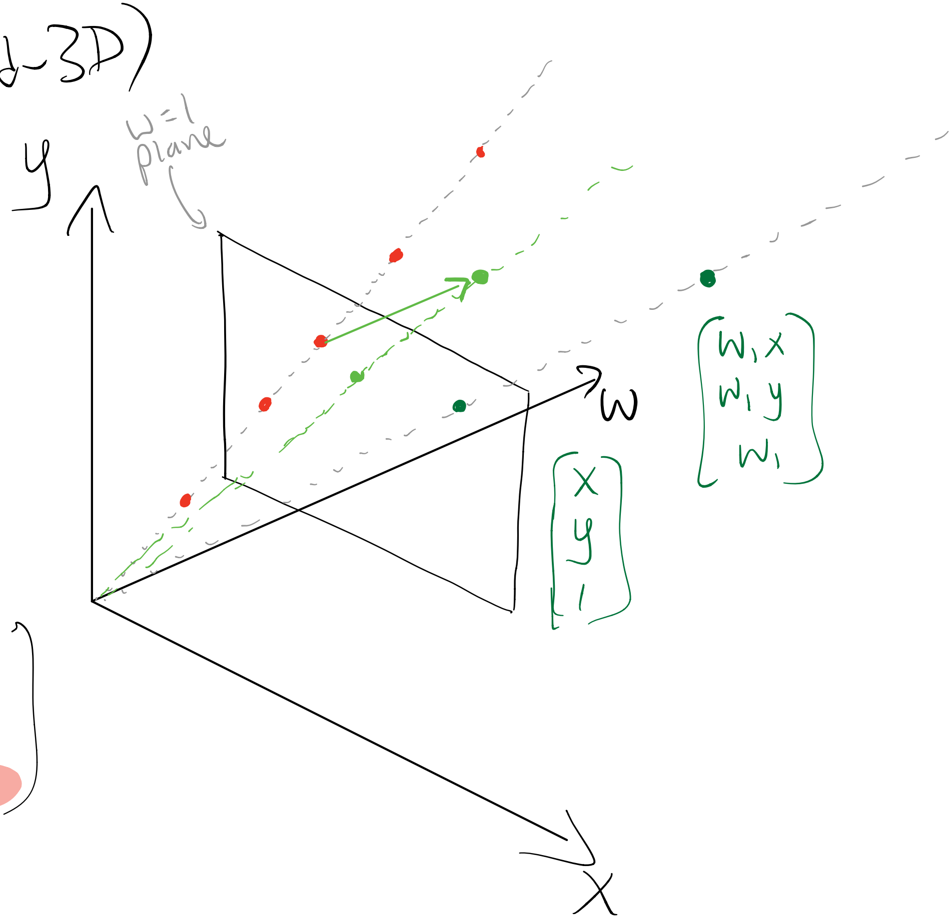
Homogeneous Coordinates: Intuition for our math hack

A 3-vector belongs to a **family** of 3-vectors representing the same 2D point.

1D example, for intuition: A 2-vector belongs to a **family** of 2-vectors representing the same 1D point.



In 2D: (Pretend-3D)



Affine:

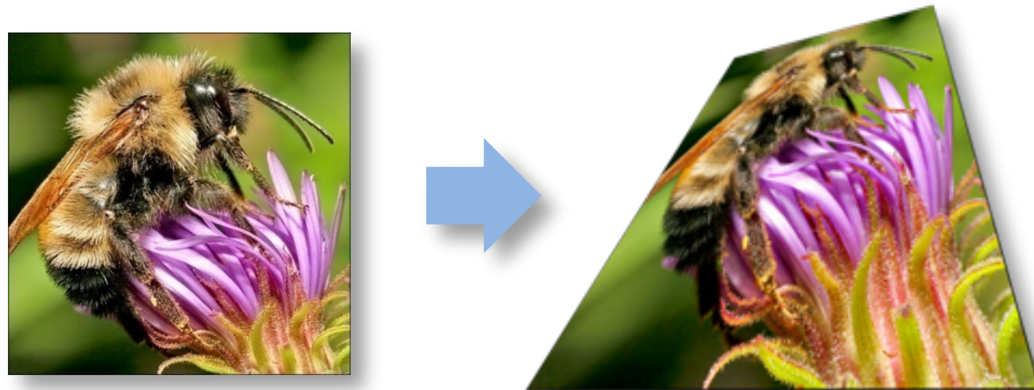
$$\begin{bmatrix} \; \\ \; \\ 1 \end{bmatrix} = \begin{bmatrix} a & b & c \\ d & e & f \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

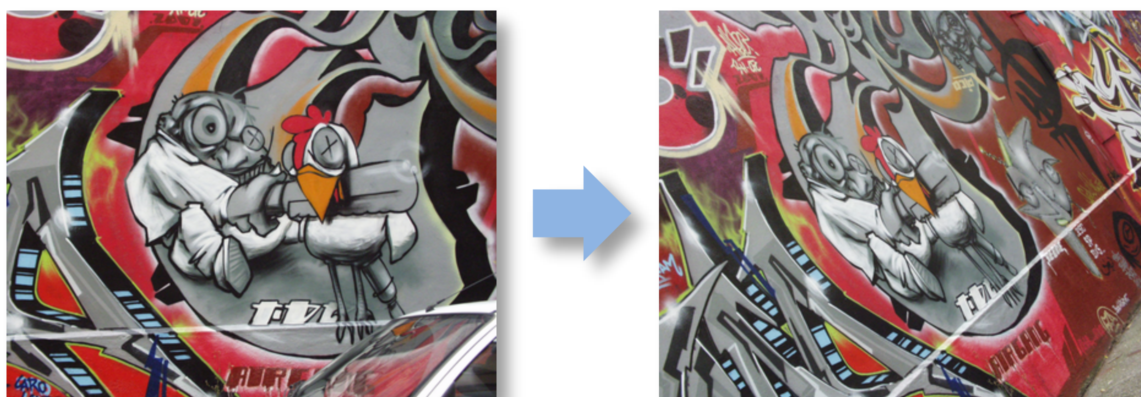
Homography:

$$\begin{bmatrix} \; \\ \; \\ \square \end{bmatrix} \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

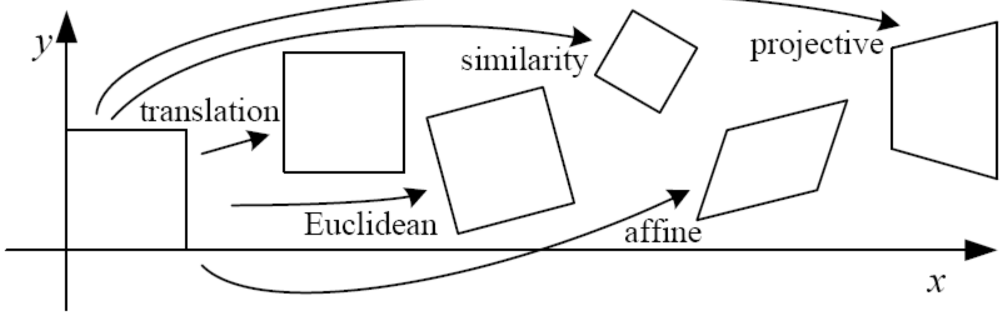
What are we looking at?

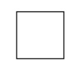
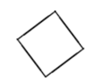
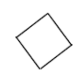
$$\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & 1 \end{bmatrix}$$





Transformations: A Hierarchy



Name	Matrix	# D.O.F.	Preserves:	Icon
translation	$\begin{bmatrix} \boldsymbol{I} \mid \boldsymbol{t} \end{bmatrix}_{2\times3}$	2	orientation $+ \cdots$	
rigid (Euclidean)	$\begin{bmatrix} \boldsymbol{R} \mid \boldsymbol{t} \end{bmatrix}_{2\times3}$	3	lengths $+ \cdots$	
similarity	$\begin{bmatrix} s\boldsymbol{R} \mid \boldsymbol{t} \end{bmatrix}_{2\times3}$	4	angles $+ \cdots$	
affine	$\begin{bmatrix} \boldsymbol{A} \end{bmatrix}_{2\times3}$	6	parallelism $+ \cdots$	
projective	$\begin{bmatrix} \tilde{\boldsymbol{H}} \end{bmatrix}_{3\times3}$	8	straight lines	

Homographies for image alignment

Why the 1 in the bottom right?

$$\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & 1 \end{bmatrix}$$

Prove on HW2: For any 3x3 matrix *H*, there exists another 3x3 matrix *H'* which:

- Has a 1 in the bottom-right corner
- Has the same effect on homogeneous points.

Why the 1 in the bottom right?

$$\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & 1 \end{bmatrix}$$

Prove on HW2: For any 3x3 matrix *H*, there exists another 3x3 matrix *H'* which:

- Has a 1 in the bottom-right corner
- Has the same effect on homogeneous points.

Fixing the 1 is merely a convention.

Another convention: values in H as a vector have magnitude 1.

Why the 1 in the bottom right?

$$\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & 1 \end{bmatrix}$$

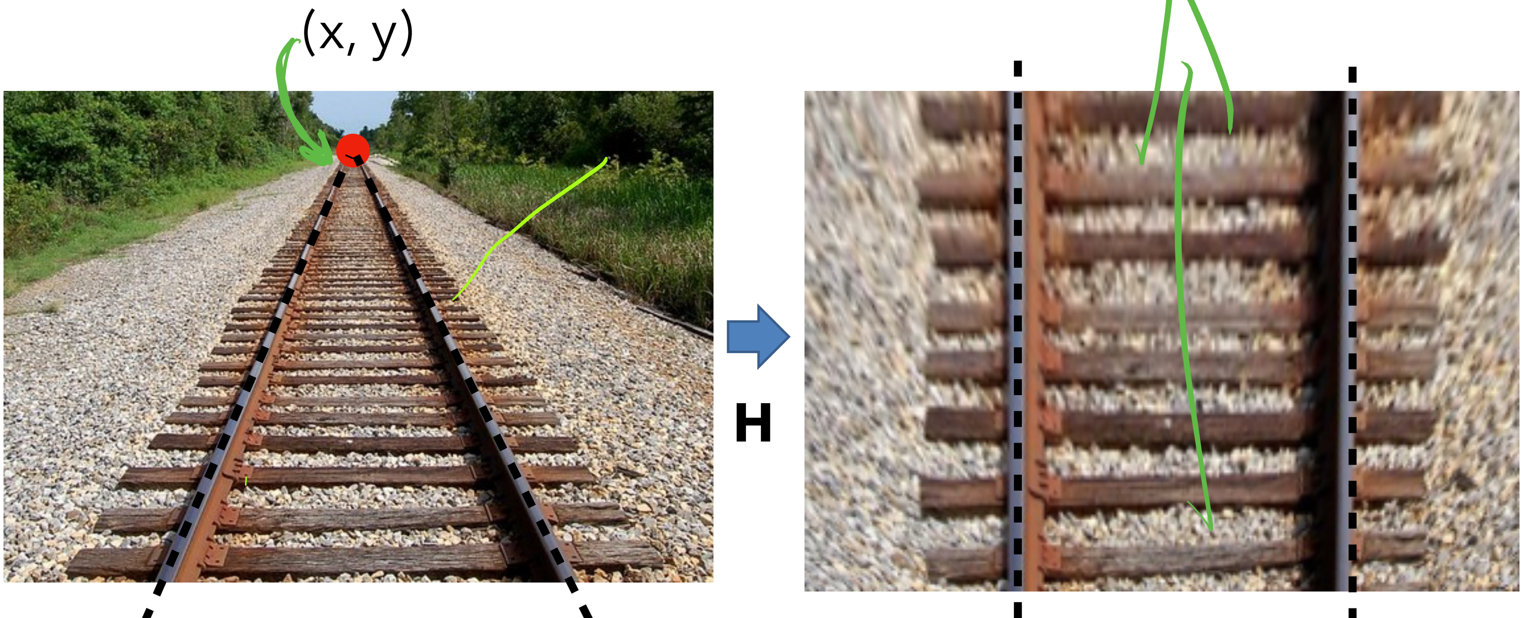
Prove on HW2: For any 3x3 matrix *H*, there exists another 3x3 matrix *H'* which:

- Has a 1 in the bottom-right corner
- Has the same effect on homogeneous points.

Consequently:

- homographies have 8 degrees of freedom (DOF).
- affine transformations have 6 DOF
- translations have 2 DOF

About those parallel lines...



The homography H transforms points from the left image to the right image.

What is $H\begin{bmatrix}x\\y\\1\end{bmatrix}$?

not a point → $\begin{bmatrix}x\\y\\0\end{bmatrix}$

point at infinity

$\begin{bmatrix}0\\1\\0\end{bmatrix}$

point at infinity

Next up:

(imperfect)

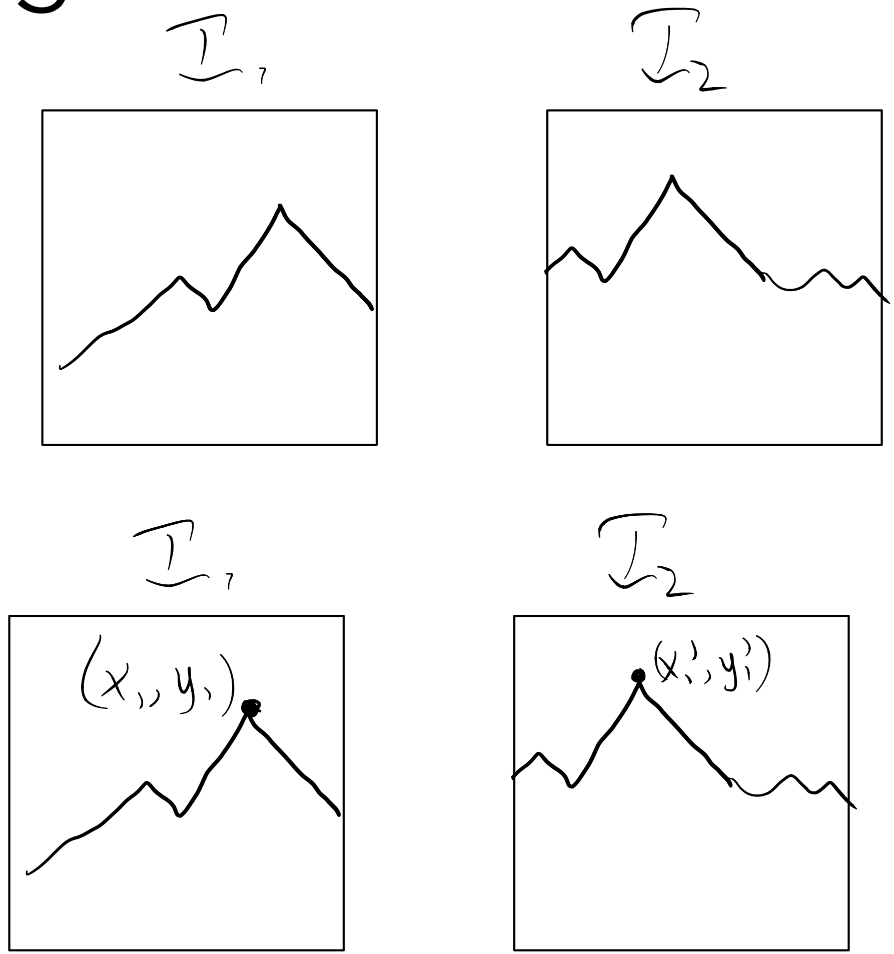
Given a set of feature matches, how do I **find** the **transformation** that relates the two images?

Image Alignment: Translation

$$t_x = x_1' - x_1$$

$$t_y = y_1' - y_1$$

$$\begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{bmatrix}$$



Mutiple Matches?

$$t_x = \frac{1}{n}\sum_{i=0}^{n}(x_i' - x_i)$$

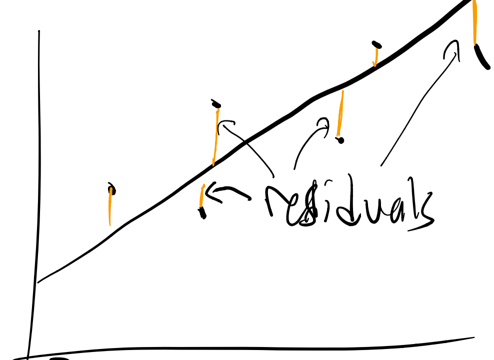
$$t_y = \frac{1}{n}\sum_{i=0}^{n}(y_i' - y_i)$$

Multiple Matches: Linear Algebra Edition!

Goal: Minimize difference between observed landing pt and model's prediction:

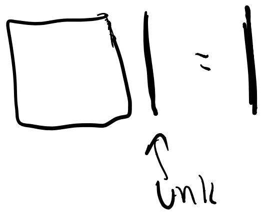
$$\sum_{i=0}^{n}\left(\begin{bmatrix} x_i' \\ y_i' \end{bmatrix} - \begin{bmatrix} x_i + t_x \\ y_i + t_y \end{bmatrix}\right)^2$$

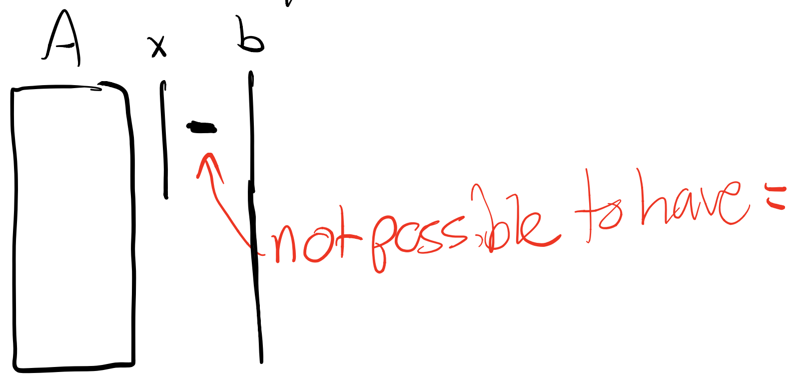
$$\min_x \|Ax - b\|^2$$



$\|\|\|\|^2$

$Ax = b$





For Translation:

- $r_i(t_x)^2 = \left(x_i' - (x_i + t_x)\right)^2$
- $r_i(t_y)^2 = \left(y_i' - (y_i + t_y)\right)^2$
-
-

$\Big\}\ 2n$

$$\left(t_x - (x_i' - x_i)\right)^2$$

unknowns → x

$$\min_{x} \; \underset{2n\times 2}{A} \; \underset{2\times 1}{x} - \underset{2n\times 1}{b}$$

$$\underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \\ 0 & 1 \\ \vdots & \vdots \end{bmatrix}}_{2n} \begin{bmatrix} t_x \\ t_y \end{bmatrix} - \begin{bmatrix} x_1' - x_1 \\ y_1' - y_1 \\ x_2' - x_2 \\ y_2' - y_2 \\ \vdots \end{bmatrix}$$

$$t_x - (x_1' - x_1)$$

$$t_y - (y_1' - y_1)$$

Solving $\min_x \|Ax - b\|^2$

Math class: normal equations

$$(A^T A)^{-1}(A^T A)x = A^T b$$

$$\boxed{x = (A^T A)^{-1} A^T b}$$

On a computer:

```
np.linalg.lstsq(A, b)
```

← use this!