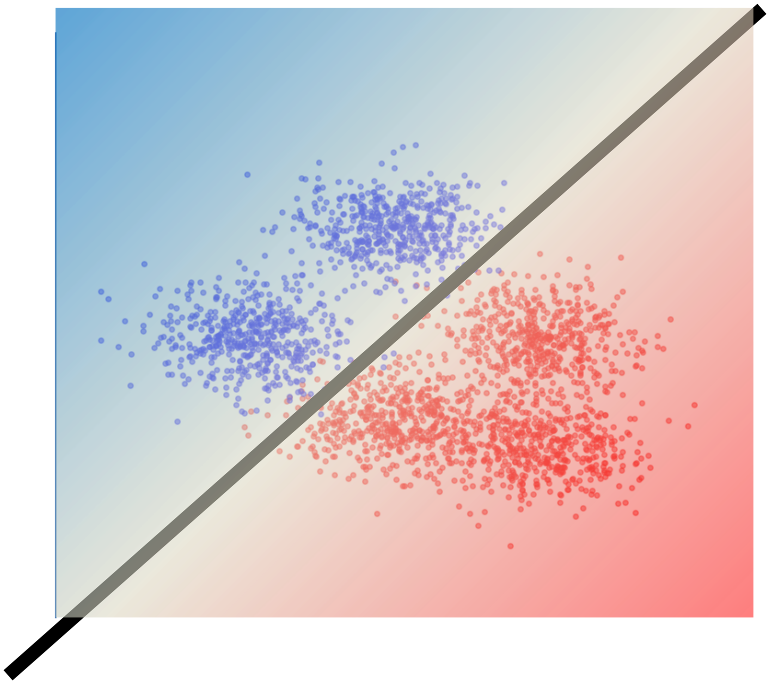
CSCI 497P/597P: Computer Vision

Linear Classifiers
(Stochastic) Gradient Descent



Readings

with a great deal more detail…

- https://cs231n.github.io/linear-classify/
- http://cs231n.github.io/optimization-1/
- http://cs231n.github.io/optimization-2/

Goals

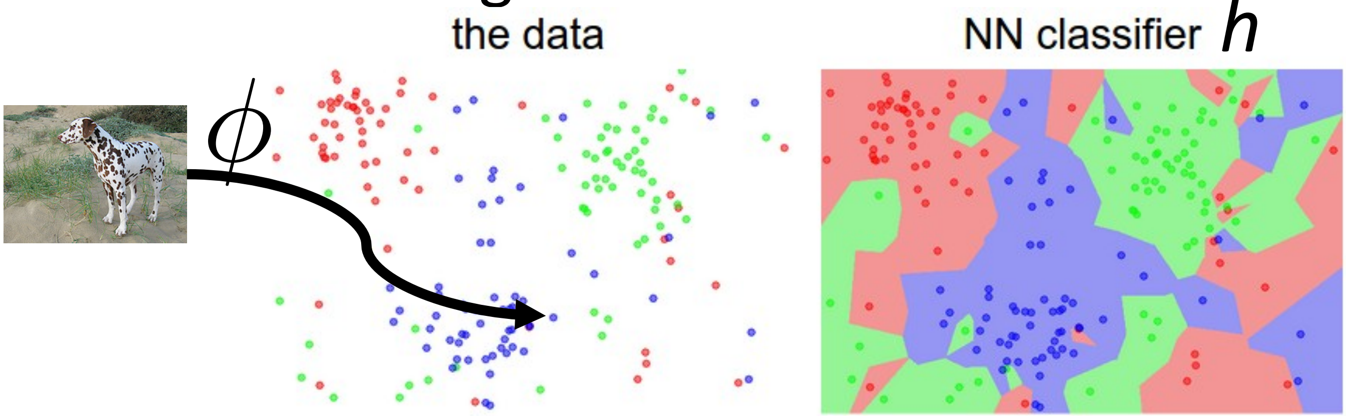
- Know the benefits and limitations of linear classifiers over KNN.
- Understand the mathematical formulation of a binary and multiclass linear classifier.
- Understand how to train a classifier by minimizing a loss function using gradient descent.
- Understand the intuition behind using Stochastic (Minibatch) Gradient Descent.

KNN: Bottom Line

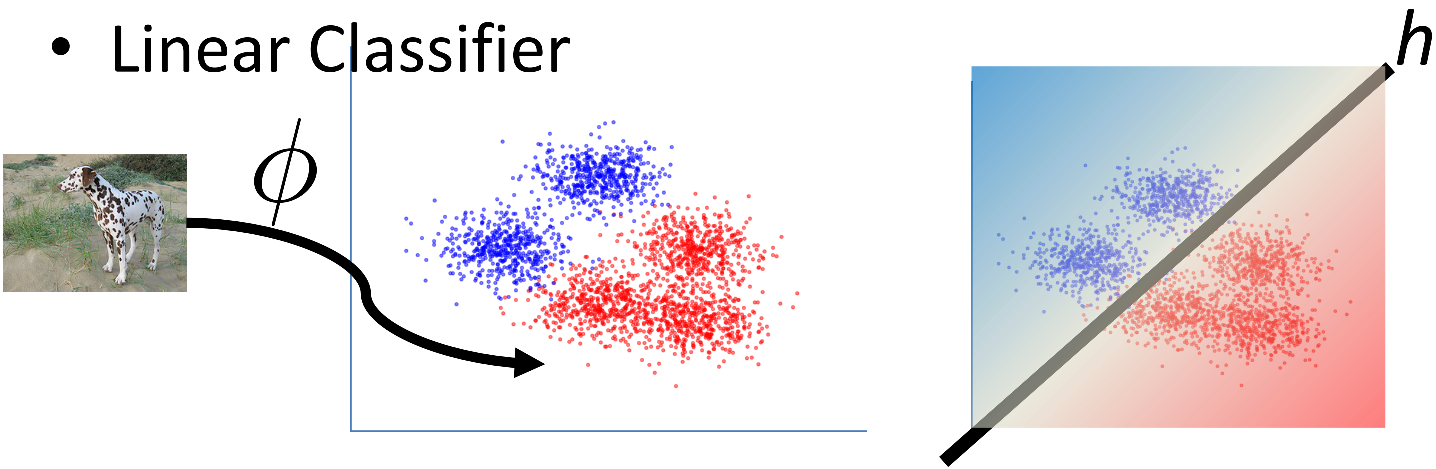
- Fast to train but slow to predict
- Distance metrics don’t behave well for high-dimensional image vectors

Classifying Images: Let's simplify

- Nearest Neighbor Classifier



- Linear Classifier

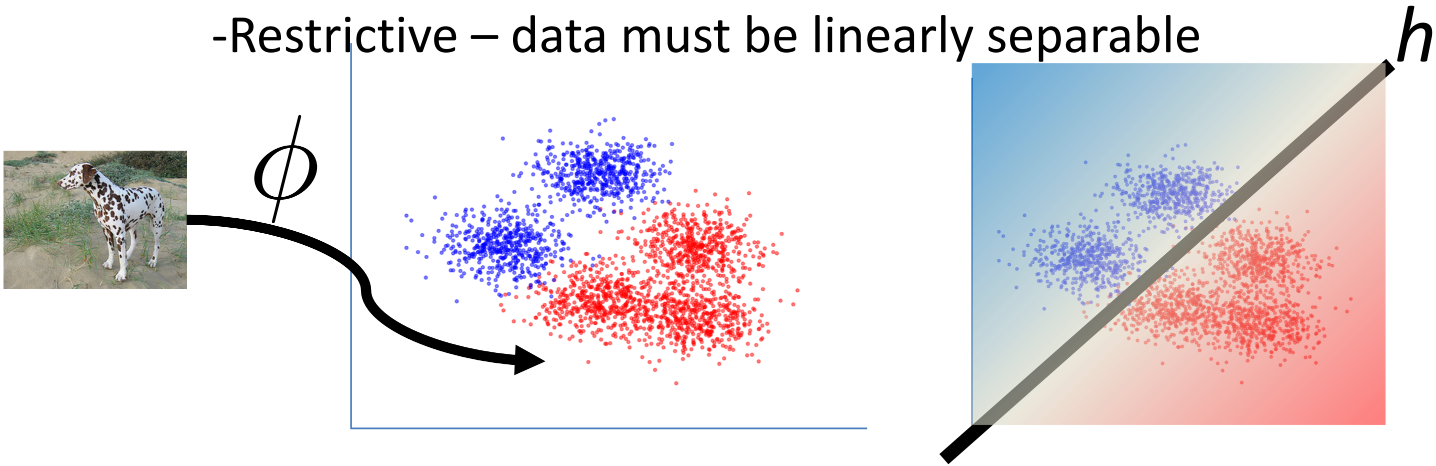


Linear classifiers

- Finding nearest neighbor is slow.
- Basic idea:
  - Training time: find a line that separates the data
  - Testing time: which side of the line is ϕ(x) on?

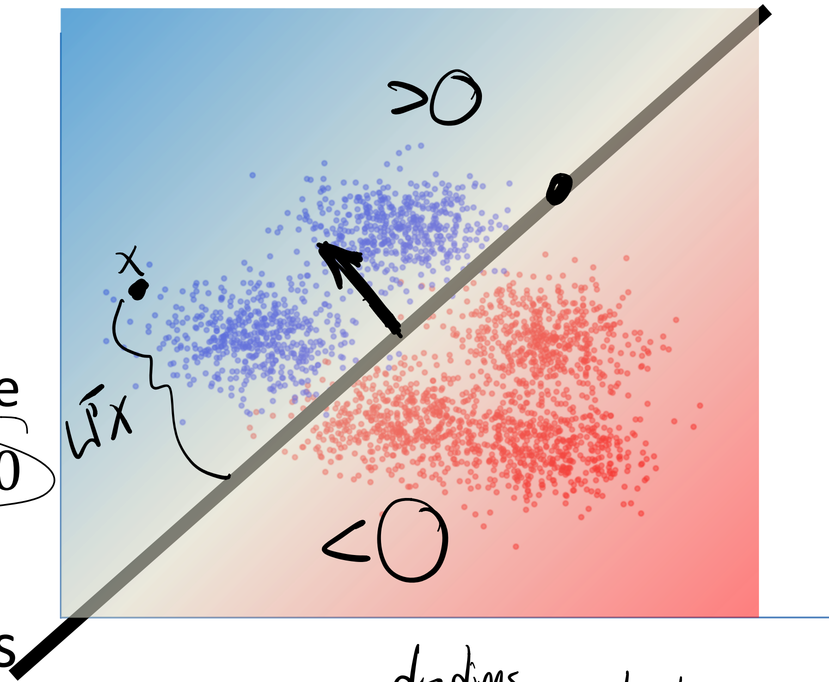
    +Fast to compute

    -Restrictive – data must be linearly separable



Linear classifiers

- A linear classifier corresponds to a hyperplane
  - Equivalent of a line in high-dimensional space
  - Equation: $w^T x + b = 0$
- Points on the same side are the same class



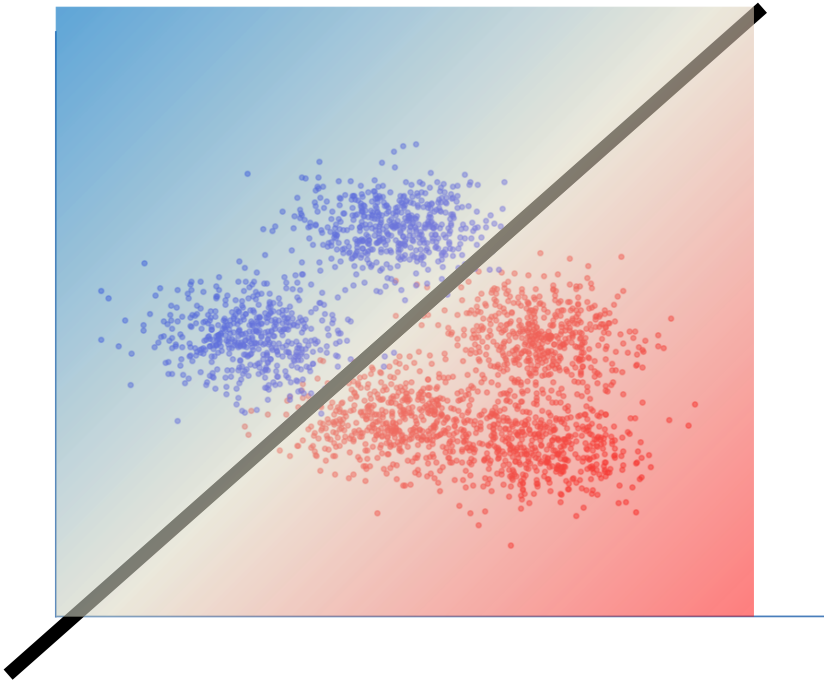
d-dims
d-dims

$$ax + by + c = 0$$

$$\overbrace{\begin{bmatrix} a & b & c \end{bmatrix}}^{w} \overbrace{\begin{pmatrix} x \\ y \\ 1 \end{pmatrix}}^{x} = 0$$

Does this ever work?

- It's easier to be linearly separable in high-dimensional space.
- But simple linear classifiers still don't work on most interesting data.



Some history from the Ante**deep**luvian Era

- Example pipeline from days of yore:
  - Detect corners and extract SIFT features
  - Collect features into a “bag of features”
  - (if you’re feeling fancy) maintain some spatial information
  - Somehow convert feature bag to fixed size
  - Apply **linear** classifier

- Key idea: ϕ is designed by hand, while *h* is learned from data.

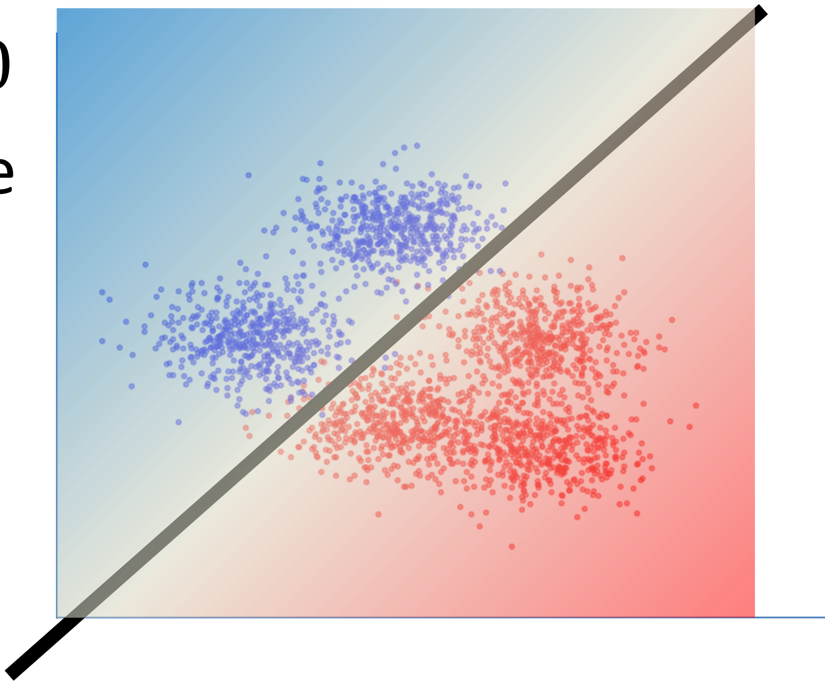
Some history of the Ante**deep**luvian Era

- Key idea: ϕ is designed by hand, while *h* is learned from data.

- Nowadays: learn both from data - "end-to-end": image goes in, label comes out.
  – Enabled only recently by bigger
    - labeled datasets
    - compute power (GPUs)

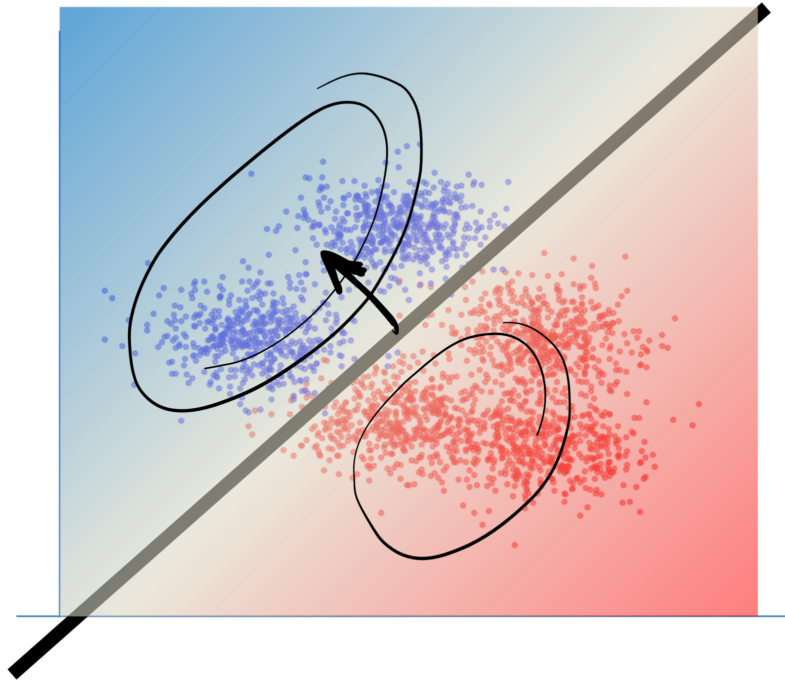
Linear classifiers

- Equation: $w^T x + b = 0$
- Points on the same side are the same class

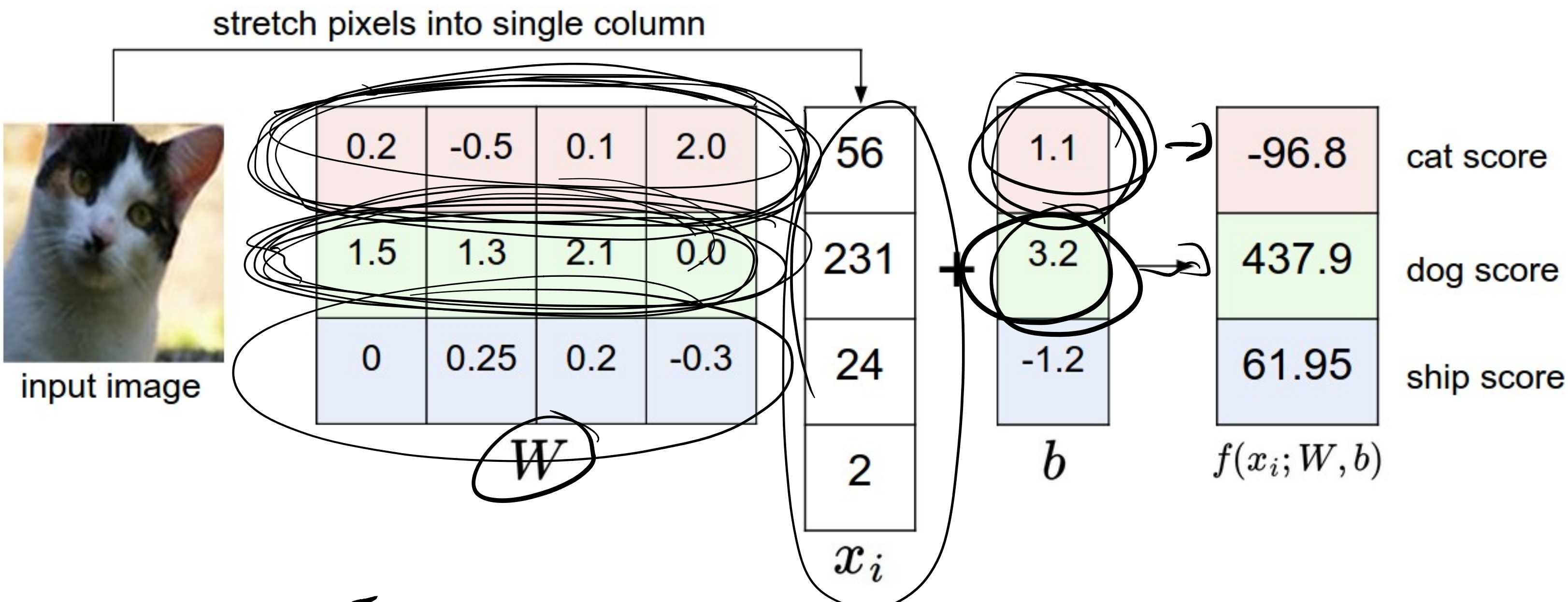


We have a classifier

- $h(x) = w^T x + b$ gives a *score*
- Score negative: red
- Score positive: blue
- Does it solve the runtime issues of KNN?



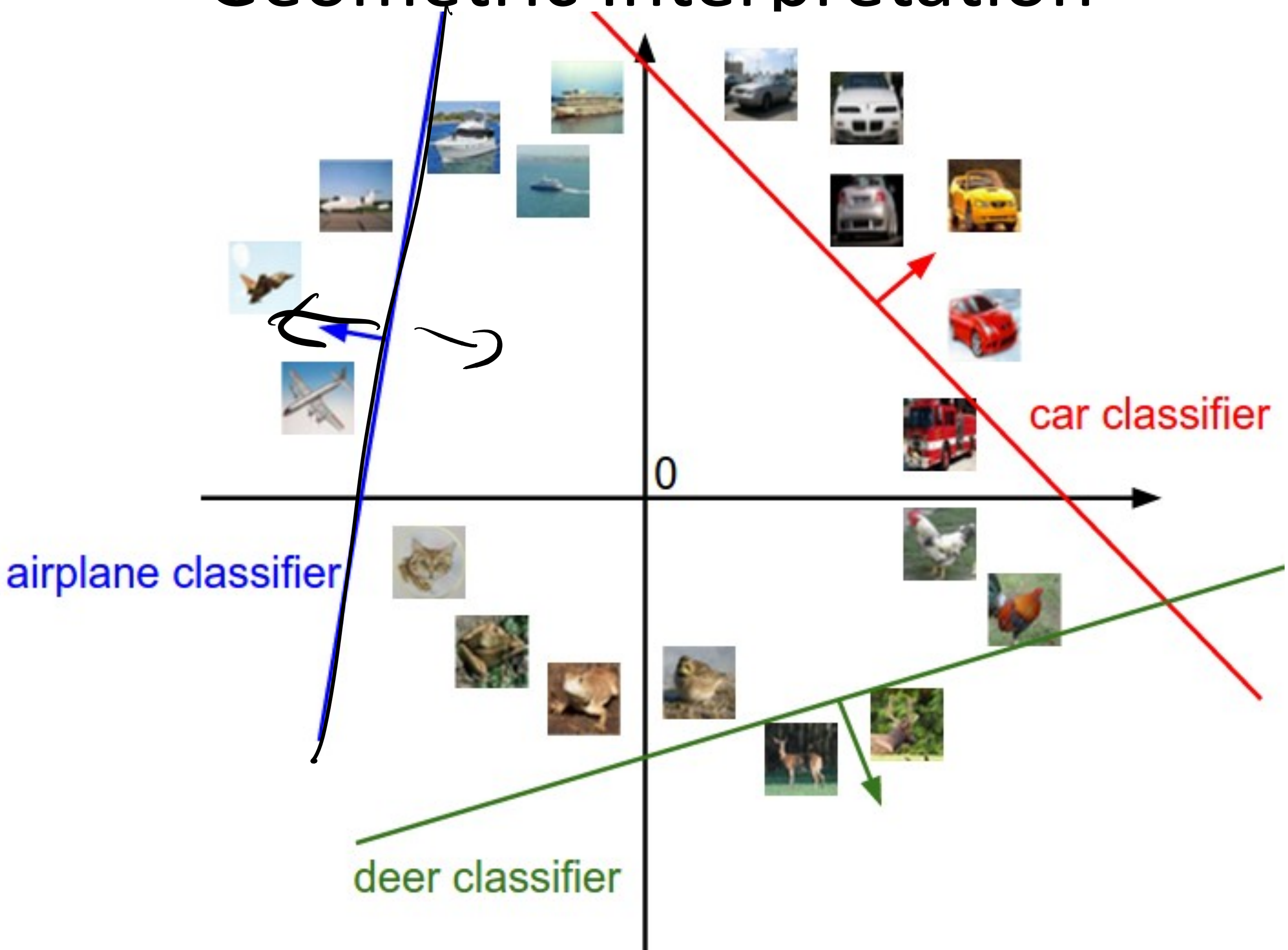
Multiclass Linear Classifiers: Stack multiple w^T into a matrix.



$\vec{w}^T x$

$W x$

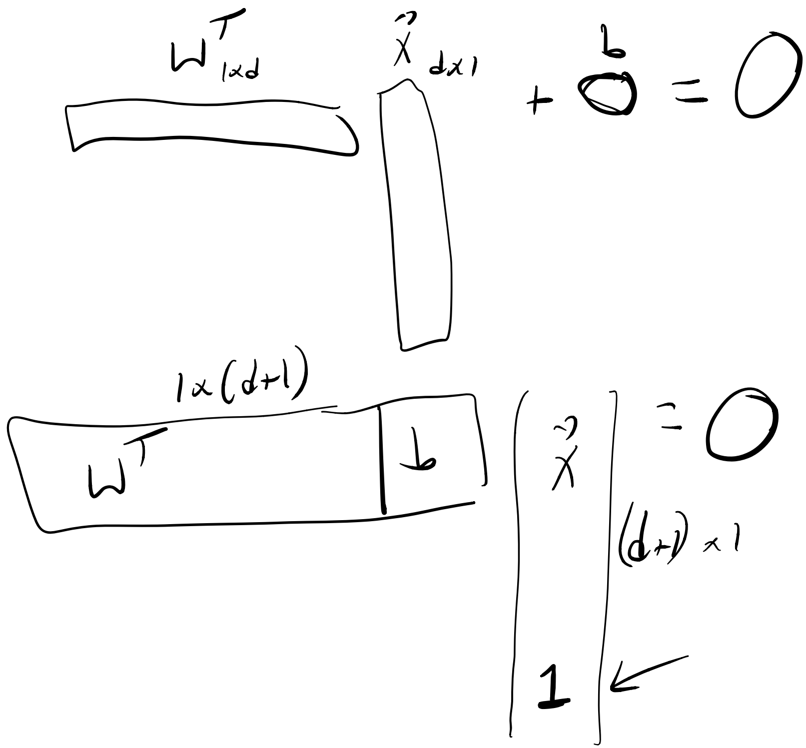
Multiclass Linear Classifier: Geometric Interpretation



The Bias Trick

augmented

$$w^T x + b = 0 \longrightarrow w^T x = 0$$



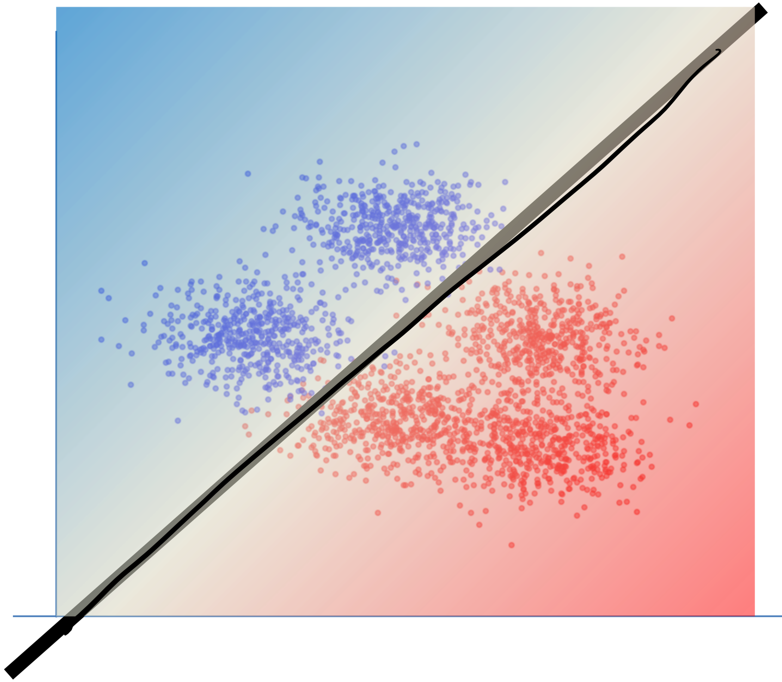
The Bias Trick

- Fold b into an additional dimension of w
- Add a fixed 1 to all feature vectors.

- Now, $h(x) = w^T x$

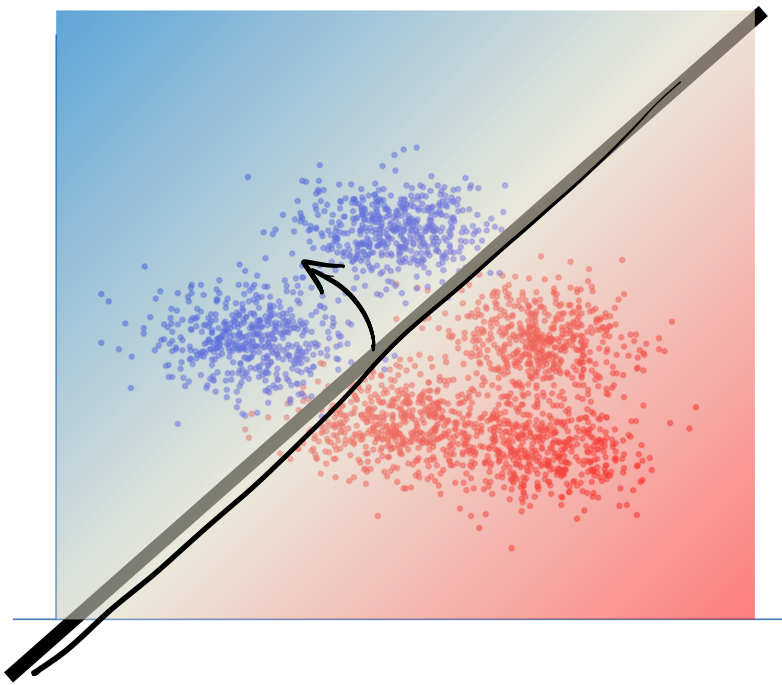
We have a classifier

- h(x) = w^T x gives a *score*
- Score negative: red
- Score positive: blue
- Where does w come from?



How do we find a good W?

- Step 1: For a given W, decide on a **Loss Function**: a measure of how much we dislike the line.
- Step 2: use **optimization** to find the W that *minimizes* the loss function.



Loss Functions

- Step 1: For a given W, decide on a **Loss Function**: a measure of how much we dislike this classifier.
- Step 2: use **optimization** to find the W that *minimizes* the loss function.
  - Linear regression: solvable in closed form
  - Useful loss functions in vision/ML: no closed form.

$$\min_x \|Ax - b\| \leftarrow$$

$$\min \|Ah\| \quad \text{s.t. } \|h\| = 1$$

Loss Functions

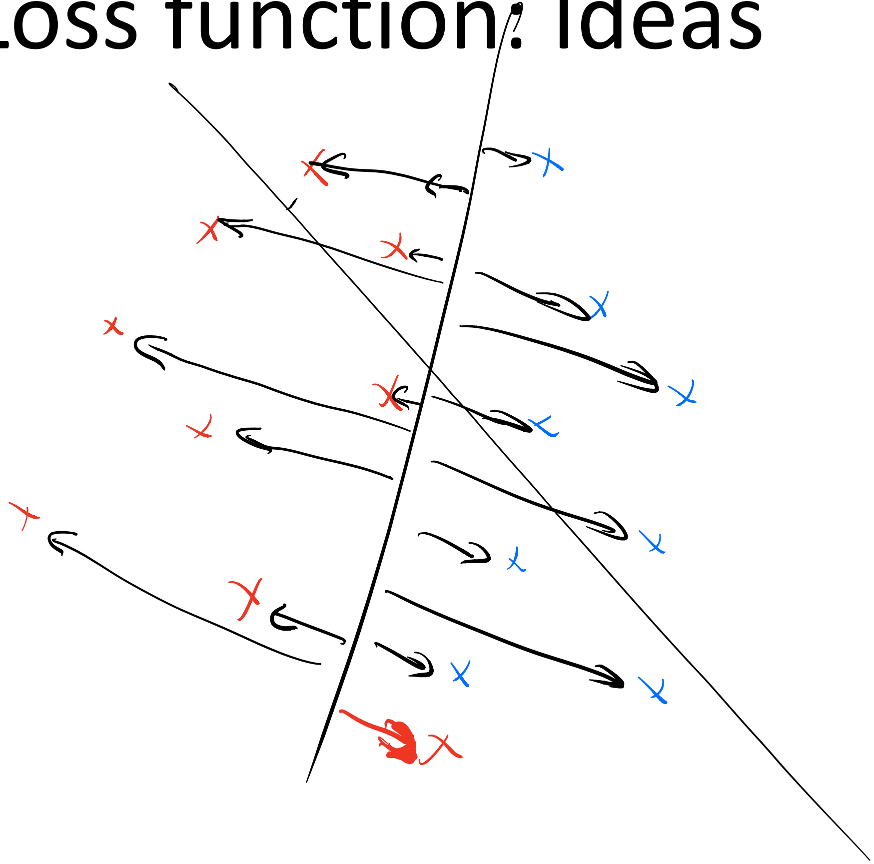
- Step 1: For a given W, decide on a **Loss Function**: a measure of how much we *dislike* this classifier.

- Loss Function intuition:
  - loss should be large if many data points are misclassified
  - loss should be small (0?) if all data is classified correctly.

Loss function: Ideas

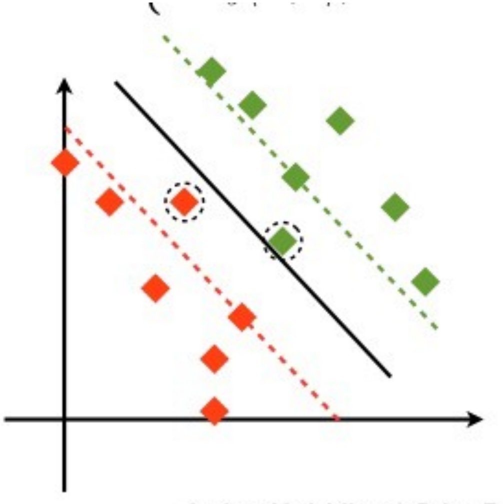
0-1 loss:

how many correct?



Loss Functions – SVM Loss

- SVM Loss:
  - Insists that data points are not just correctly classified, but a certain distance from the hyperplane:
  - $L_i = \max(0, x_i, 1- y_i(w^T x_i + b)$



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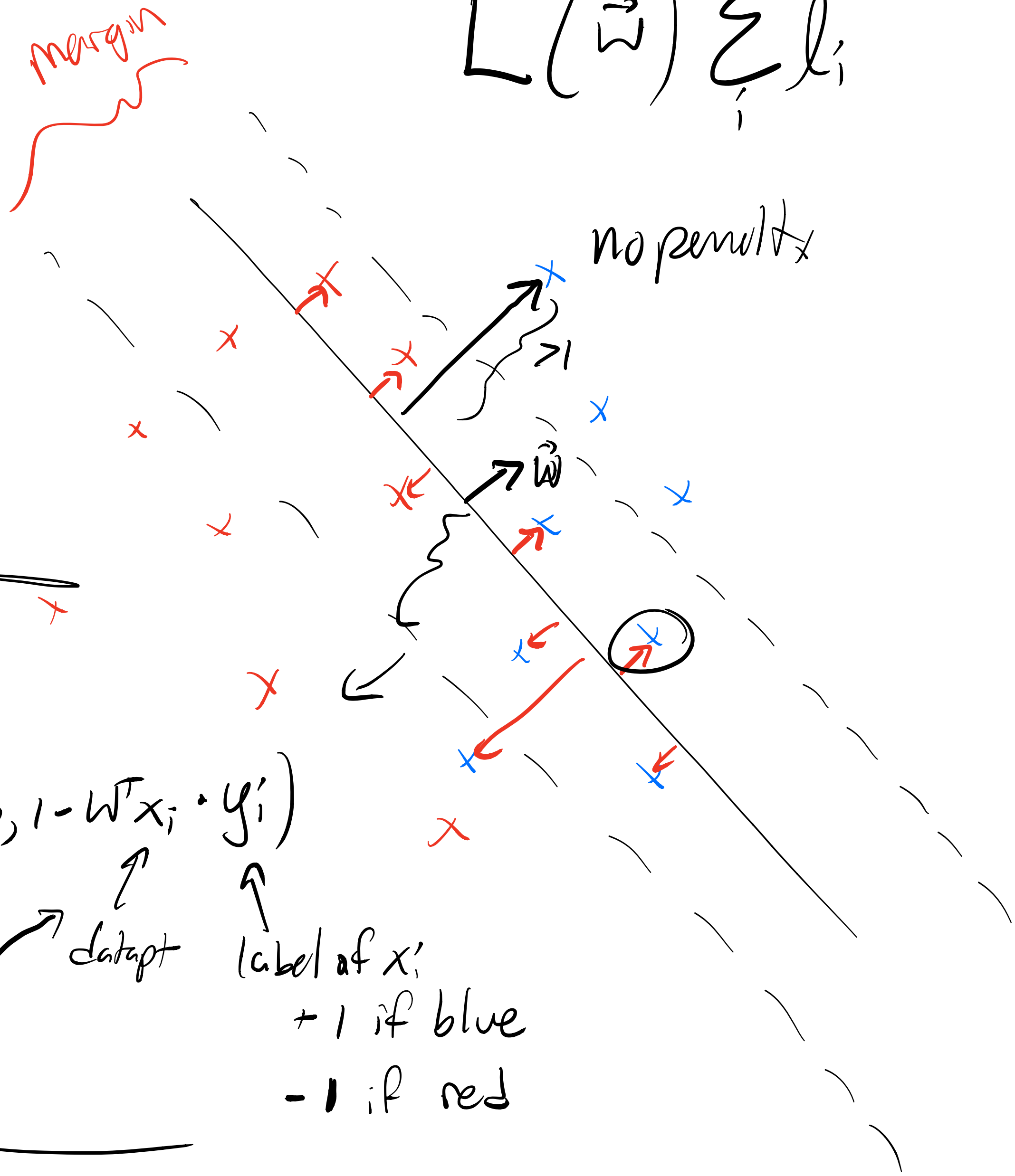
x_i = i'th data point

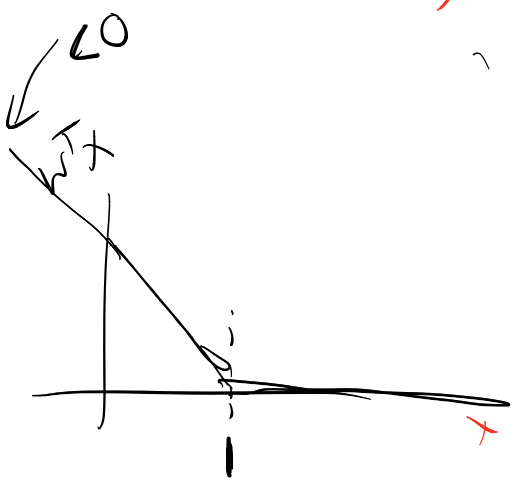
y_i = i'th data point's true label:

-1 if red

+1 if green

$L(\vec{w}) \sum_i l_i$





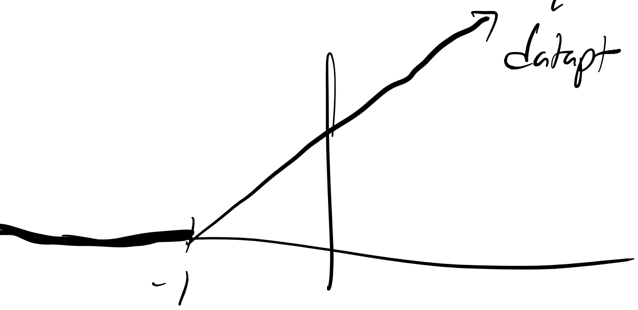
$l_i = \max(0, 1 - w^T x_i \cdot y_i)$

datapt

label of x_i

+1 if blue

-1 if red

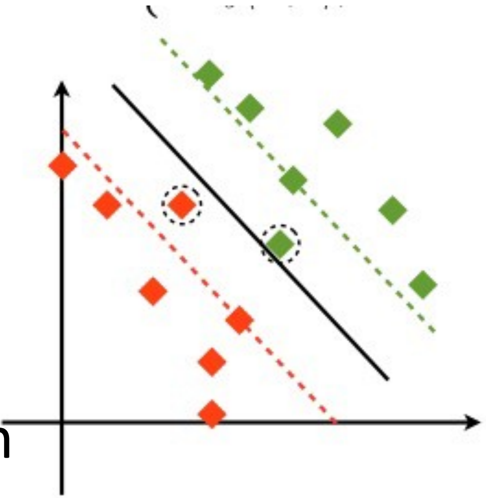


Loss Functions – SVM Loss

- SVM Loss:
  - Insists that data points are not just correctly classified, but a certain distance from the hyperplane:
  - $L_i = \max(0\ x_i, 1- y_i(w^T x_i + b)$

    x_i = i'th data point
    y_i = i'th data point's true label:
    -1 if red
    +1 if green

  - $L(w, b) = \Sigma_i L_i$
  - Loss for a given line is the sum of the loss for all datapoints



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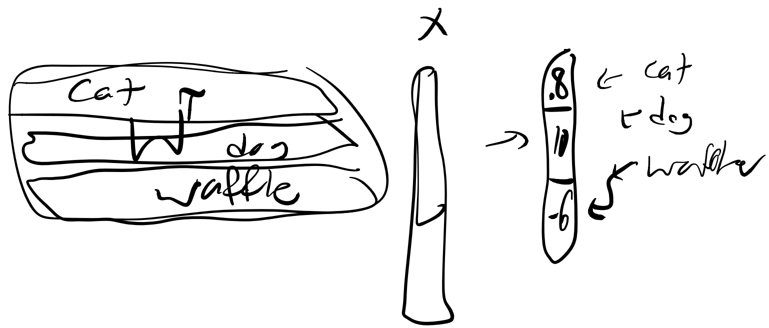
Softmax Classifier / Cross-Entropy Loss: Intuition

$W^T x$ gives us a vector of scores, one per class (each row of W is a classifier)

Wouldn't it be nice to interpret these as probabilities?



Softmax Classifier / Cross-Entropy Loss: Intuition

$W^T x$ gives us a vector of scores, one per class (each row of W is a classifier)

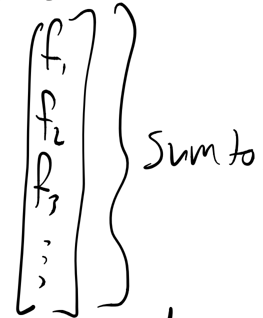
Wouldn't it be nice to interpret these as probabilities?

But they're not…

- can be < 0
- don't all sum to 1

But we can treat them as **unnormalized log probabilities**.

Softmax Classifier / Cross-Entropy Loss



f = W^T x gives us a vector of scores, one per class (each row of W is a classifier)

Softmax normalization: Exponentiate to get all positive values, then normalize to sum to 1:

$$p(x_i \text{ is class } k) = \frac{e^{f_k}}{\sum_j e^{f_j}}$$

Softmax Classifier / Cross-Entropy Loss

$f = W^T x$ gives us a vector of scores, one per class (each row of W is a classifier)

Softmax normalization: Exponentiate to get all positive values, then normalize to sum to 1:

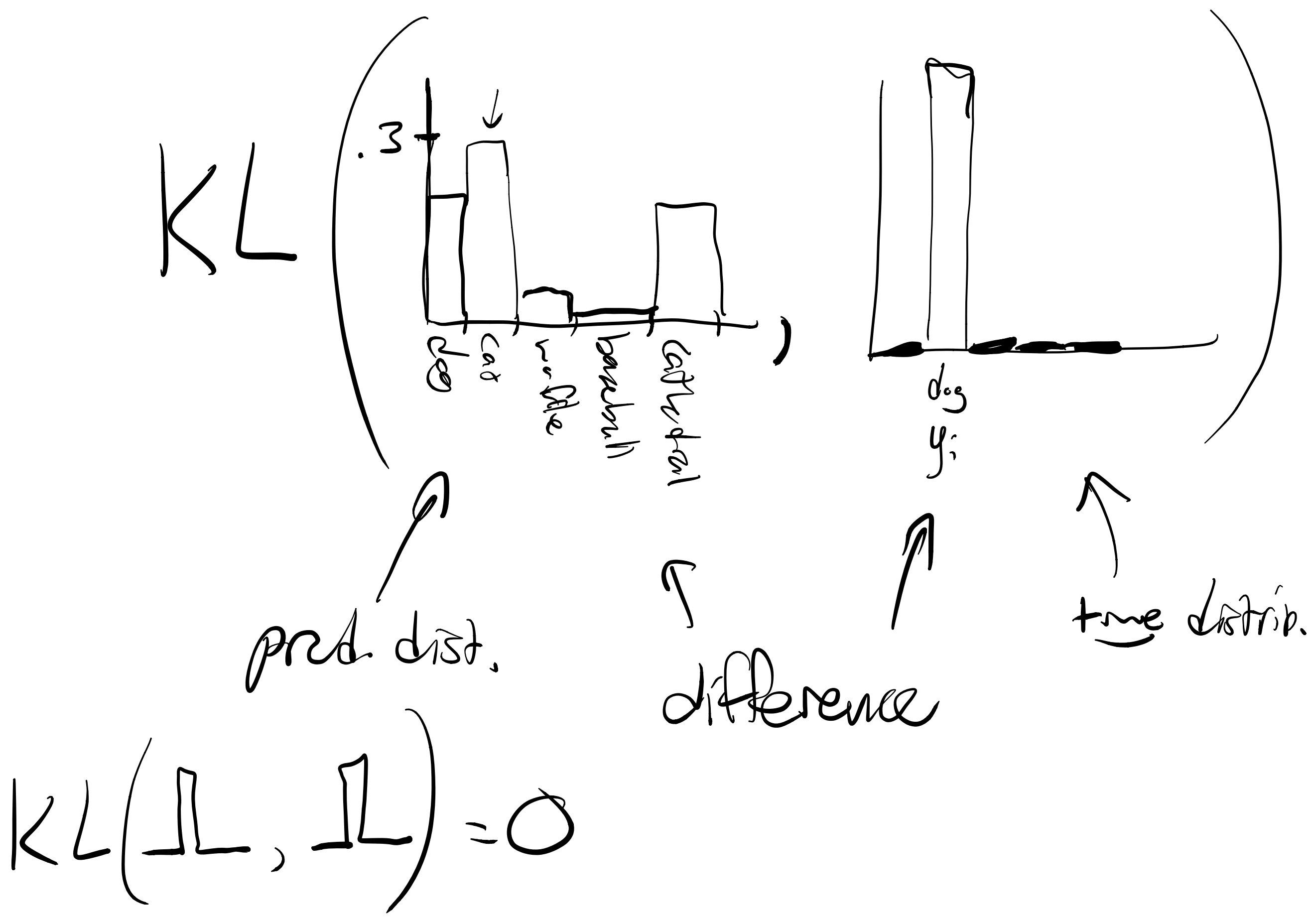
$$p(x_i \text{ is class } k) = \frac{e^{f_k}}{\sum_j e^{f_j}}$$

Cross-entropy loss: measure *KL divergence* between the **predicted** distribution and the **true** distribution:

true label

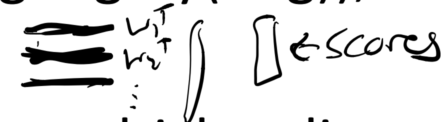
$$L_i = -\log\left(\frac{e^{f_{y_i}}}{\sum_j e^{f_j}}\right)$$

Cross-Entropy Loss: Intuition



Taking stock

- We have:
  - φ = unravel(rgb2gray(img)), a feature extractor



  - h(x) = W^Tx, a multiclass linear classifier
  - L = $\sum_{i=0}^{N} L_i$, a loss function

$$L_i = -\log\left(\frac{e^{f_{y_i}}}{\sum_j e^{f_j}}\right)$$

- We don't have:
  - a way to find a W that results in a small L.

Minimizing the Loss

- Use **optimization** to find the W that *minimizes* the loss function.
  - Linear regression: solvable in closed form
  - Most of the time: no closed form.

Optimization

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How do we find a W that minimizes L?

- Bad idea: Random search.

```
# assume X_train is the data where each column is an example (e.g. 3073 x 50,000)
# assume Y_train are the labels (e.g. 1D array of 50,000)
# assume the function L evaluates the loss function

bestloss = float("inf") # Python assigns the highest possible float value
for num in xrange(1000):
  W = np.random.randn(10, 3073) * 0.0001 # generate random parameters
  loss = L(X_train, Y_train, W) # get the loss over the entire training set
  if loss < bestloss: # keep track of the best solution
    bestloss = loss
    bestW = W
  print 'in attempt %d the loss was %f, best %f' % (num, loss, bestloss)

# prints:
# in attempt 0 the loss was 9.401632, best 9.401632
# in attempt 1 the loss was 8.959668, best 8.959668
# in attempt 2 the loss was 9.044034, best 8.959668
# in attempt 3 the loss was 9.278948, best 8.959668
# in attempt 4 the loss was 8.857370, best 8.857370
# in attempt 5 the loss was 8.943151, best 8.857370
# in attempt 6 the loss was 8.605604, best 8.605604
# ... (trunctated: continues for 1000 lines)
```

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How'd that go for you?

Lets see how well this works on the test set...

```
# Assume X_test is [3073 x 10000], Y_test [10000 x 1]
scores = Wbest.dot(Xte_cols) # 10 x 10000, the class scores for all test examples
# find the index with max score in each column (the predicted class)
Yte_predict = np.argmax(scores, axis = 0)
# and calculate accuracy (fraction of predictions that are correct)
np.mean(Yte_predict == Yte)
# returns 0.1555
```

15.5% accuracy! not bad!
(SOTA is ~95%)

Finding a W that minimizes L

- Simple idea: walk downhill.

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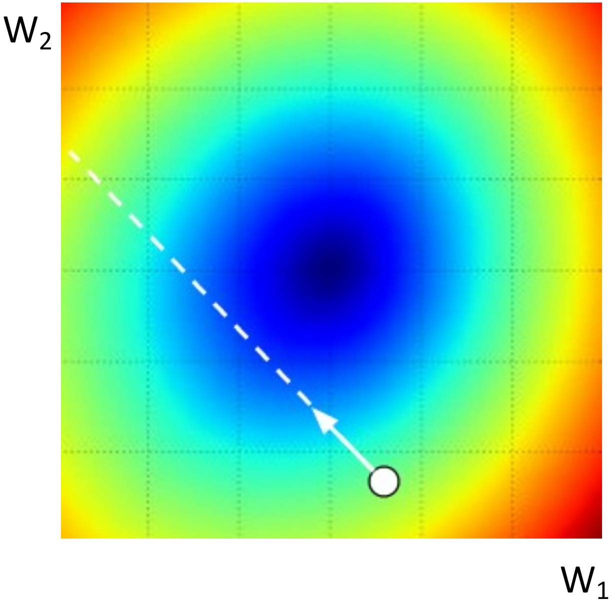
Gradient Descent: Generally

- Gradient of the loss function with respect to the *weights* tells us how to change the weights to improve the loss.



Gradient Descent: Intuition

The effect of Step Size

Too large: unstable

Too small: slow convergence

Reality isn’t quite so pretty

- Loss functions are rarely *convex*. Finding a **local minimum** is the best you can do.

Gradient Descent

```
# Vanilla Gradient Descent

while True:
  weights_grad = evaluate_gradient(loss_fun, data, weights)
  weights += - step_size * weights_grad # perform parameter update
```

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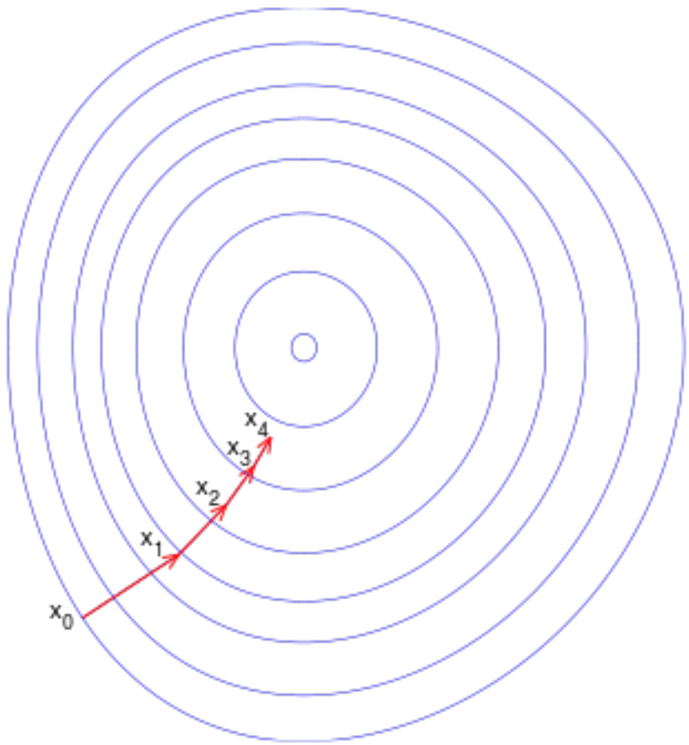
Gradient Descent: Intuition



Gradient Descent: Demo

- http://vision.stanford.edu/teaching/cs231n-demos/linear-classify/
  - select “Softmax” radio button at the bottom

Stochastic Gradient Descent

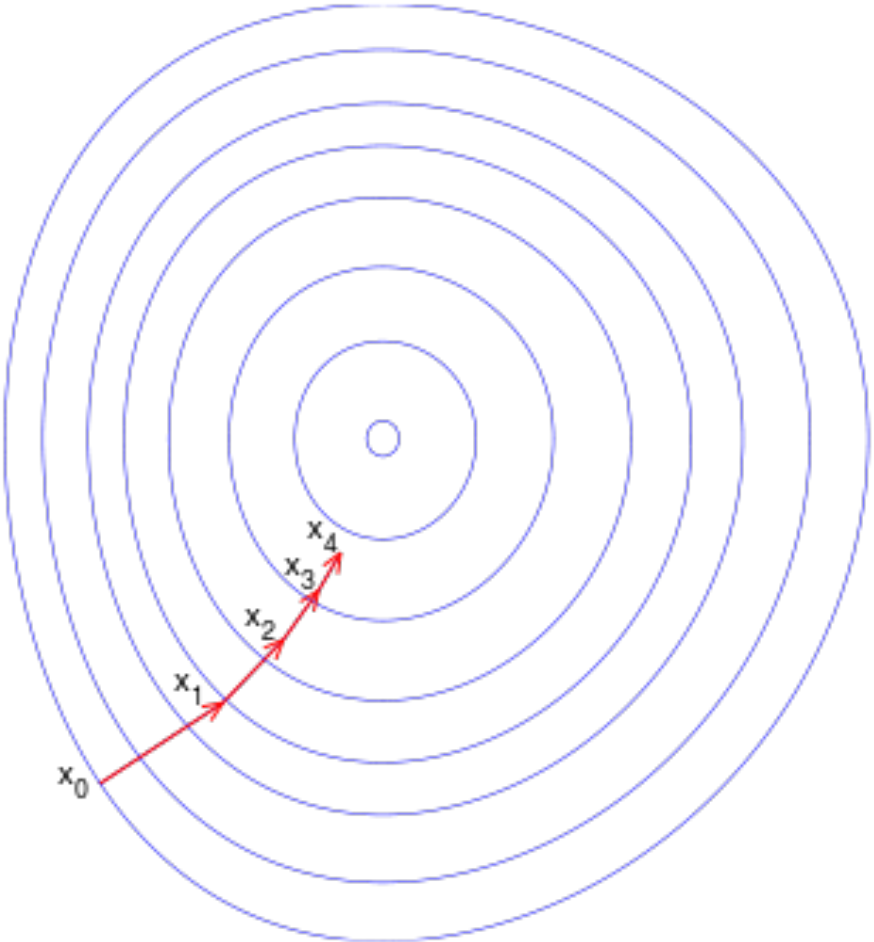
```
# Vanilla Minibatch Gradient Descent

while True:
  data_batch = sample_training_data(data, 256) # sample 256 examples
  weights_grad = evaluate_gradient(loss_fun, data_batch, weights)
  weights += - step_size * weights_grad # perform parameter update
```

- L(X, Y; W) depends on
  - All data points $x_1..x_n$
  - Ground truth labels $y_1..y_n$
  - Weights W
- Very expensive to evaluate if you have a lot of data.

Stochastic Gradient Descent

- Idea: consider only a few data points at a time.
- Loss is now computed using only a small batch (minibatch) of data points.
- Update weights the same way using the gradient of L wrt the weights.

Stochastic Gradient Descent: Intuition



Taking stock

- We have:
  - ϕ = unravel(rgb2gray(img)), a feature extractor
  - h(x) = W^T x, a multiclass linear classifier
  - L = $\sum_{i=0}^{N} L_i$, a loss function $L_i = -\log\left(\frac{e^{f_{y_i}}}{\sum_j e^{f_j}}\right)$
  - A way too adjust W until we can't make L any smaller.