Lecture 24 - Projective Geometry I

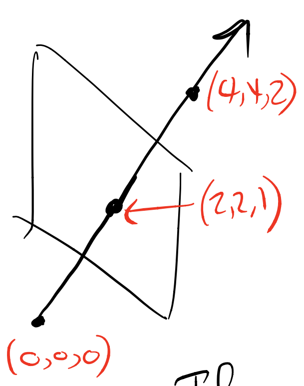
Announcements:

- P3 is out
  - (optionally) pair up by EOD Tuesday
  - due Monday, Nov 16. Start now!
- HW 4 coming soon... ~Friday

Goals:

- Know how to represent points and lines in $\mathbb{P}^2$
- Understand the duality of points and lines
  - line through two points / intersection of 2 lines
  - whether a point lies on a line / line goes through a point

Homogeneous Points (Review)



$p \in \mathbb{P}^2$ is a 3-vector representing a 2D point

$$\left\{ \begin{bmatrix} x \\ y \end{bmatrix} \xrightarrow{\text{homogenize}} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} \; ; \; \begin{bmatrix} x \\ y \\ w \end{bmatrix} \overset{\text{normalize}}{\sim} \begin{bmatrix} x/w \\ y/w \\ 1 \end{bmatrix} \xrightarrow{\text{dehomogenize}} \begin{bmatrix} x/w \\ y/w \end{bmatrix} \right.$$

If we pretend it's $\mathbb{R}^3$, then all points on the ray from $(0,0,0)$ through $\begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$ are equivalent, i.e. project (normalize) to the same place.

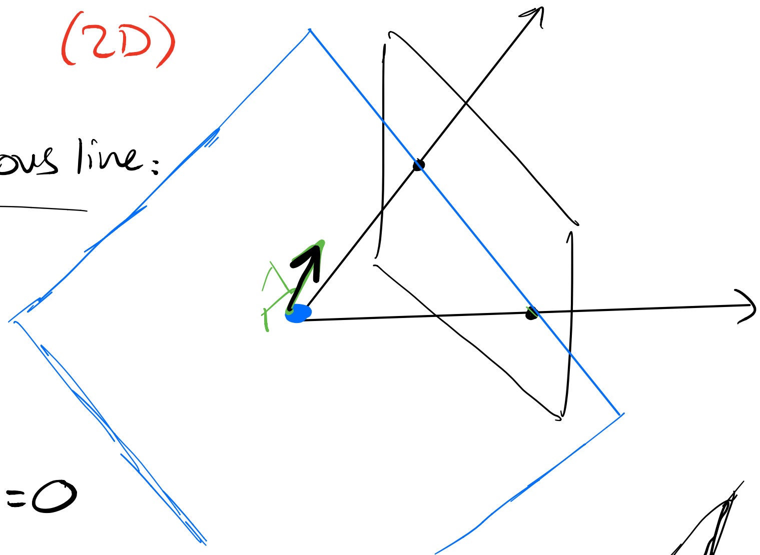
Homogeneous Lines

(0D) (1D)

A point in $\mathbb{P}^2$ is a ray in $\mathbb{R}^3$

A line in $\mathbb{P}^2$ is a plane in $\mathbb{R}^3$, through the origin

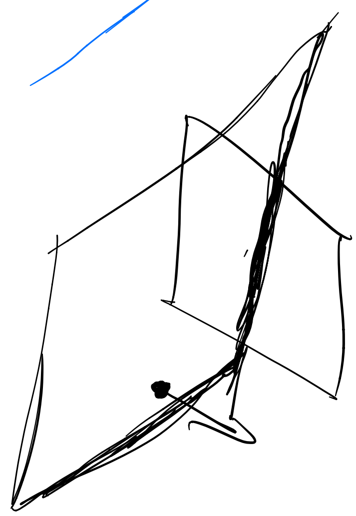
(1D) (2D)



Coordinates of a homogeneous line:

A homog. line is represented by its normal vector.

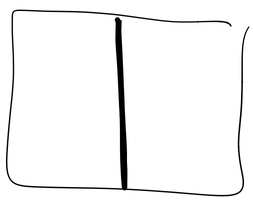
Line $\begin{bmatrix} a \\ b \\ c \end{bmatrix}$ in $\mathbb{R}^3$ projects to $ax+by+c=0$



Examples:

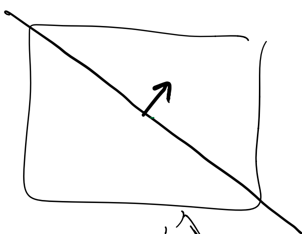
$\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ $1x+0y+0=0$

$x=0$



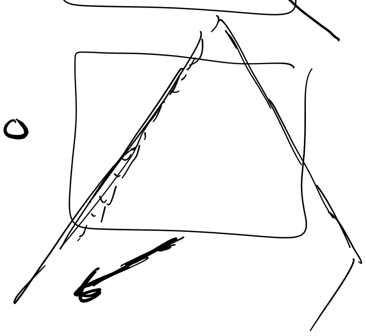
$\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ $y=-x$

$x+y=0$



$\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \\ -4 \end{bmatrix}$ $y=2x+4$

$-2x+1y-4=0$

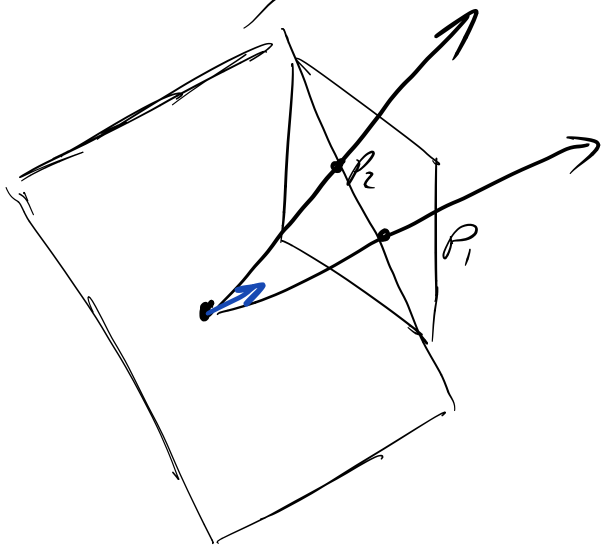


Notice: Scale Ambiguity!

$\begin{bmatrix} a \\ b \\ c \end{bmatrix} \simeq \begin{bmatrix} ka \\ kb \\ kc \end{bmatrix}$ $kax+kby+kc=0$

$(k \neq 0)$

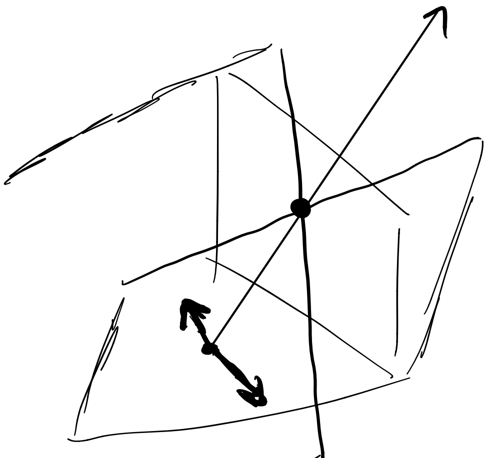
Point-Line Duality



The line through 2 points (rays) in $\mathbb{R}^3$
is the plane ($\mathbb{R}^3$) spanning those 2 rays.

The plane normal is the cross product of the 2 rays.

$$\ell_{p_1 p_2} = p_1 \times p_2$$



The point where 2 lines intersect?

The ray that lies in both planes,
i.e., is orthogonal to both plane normals.

$$p_{\ell_1 \ell_2} = \ell_1 \times \ell_2$$

Computing Cross Products

$$\begin{array}{c} p_1 \\ \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} \end{array} \times \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix} = \begin{pmatrix} y_1 z_2 - z_1 y_2 \\ z_1 x_2 - x_1 z_2 \\ x_1 y_2 - y_1 x_2 \end{pmatrix}$$ (yuck!)

Fact: This can be written as a matrix product!

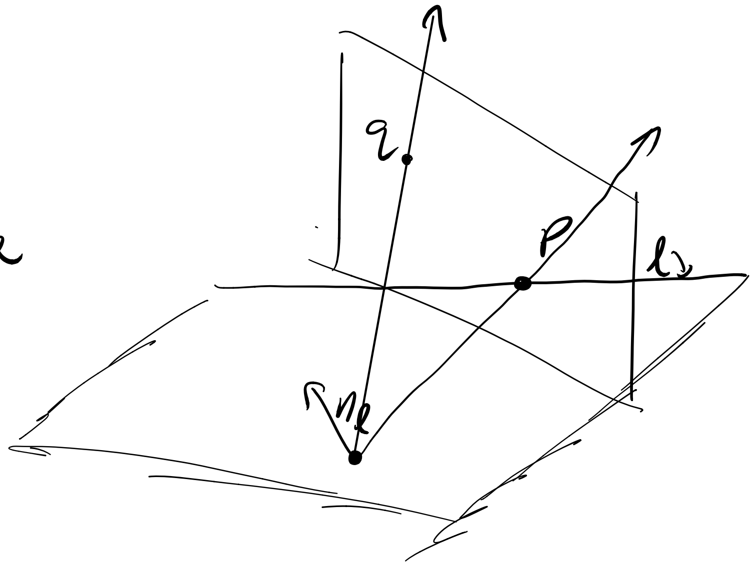
denoted:

$$\begin{bmatrix} 0 & -z_1 & y_1 \\ z_1 & 0 & -x_1 \\ -y_1 & x_1 & 0 \end{bmatrix} \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix} = p_1 \times p_2 = [p_1]_\times \cdot p_2$$

dot.

Points on Lines; Lines thru Points

If p is on l,
p's ray lies in l's plane



$p \cdot l = 0$

If l goes through p,
$l \cdot p = 0$

Algebraically:

$l = [a, b, c]^T$

$p = [x_1, y_1, z_1]^T$

$\frac{x_1}{z_1}, \frac{y_1}{z_1}, 1$

$a x_1 + b y_1 + c z_1 = 0$

$$\begin{bmatrix} a & b & c \end{bmatrix} \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} = 0$$

Next Time