Projective Geometry: Homogeneous Points

Homogeneous coordinates: math hack

homgenize: $\begin{bmatrix} x \\ y \end{bmatrix} \rightarrow \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$; dehamogenize: $\begin{bmatrix} x \\ y \\ w \end{bmatrix} \rightarrow \begin{bmatrix} x/w \\ y/w \end{bmatrix}$

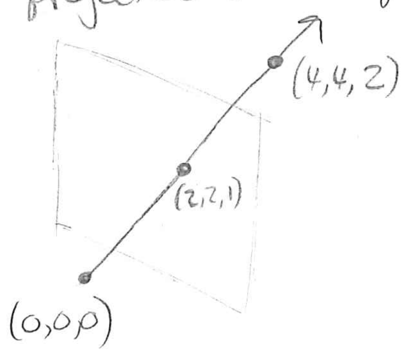
Allows us to represent translations using linear transformations (matrix multiplication).

normalize: $\begin{bmatrix} x \\ y \\ w \end{bmatrix} \rightarrow \begin{bmatrix} x/w \\ y/w \\ 1 \end{bmatrix}$

Mathematically speaking, 2D homogeneous coordinates live in

2D Projective space $\mathbb{P}^2$.

A nice geometric interpretation: objects in $\mathbb{P}^2$ are objects from $\mathbb{R}^3$, projected onto a plane using the origin $(0,0,0)$ as the COP.

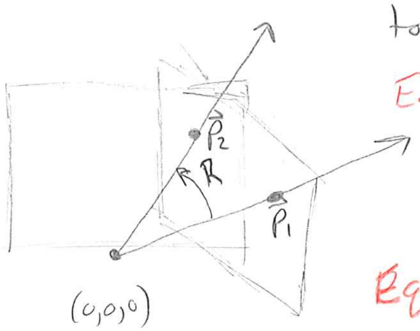


The projection means all points on the ray from $(0,0,0)$ in the direction of $\begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$ are equivalent: they project to the same point on the plane.

Interpreting Homographies

Projecting rays onto a different plane (with the same COP) is like applying a rotation in 3D to the homogeneous coordinates (eqn 1). If pixel coordinates are different from camera coordinates, i.e., $K \neq I$, we need to map from pixels to camera, rotate, then from camera to pixels (eqn 2)

rays:



Eqn 1:

$$\vec{P}_2 = R\vec{P}_1$$

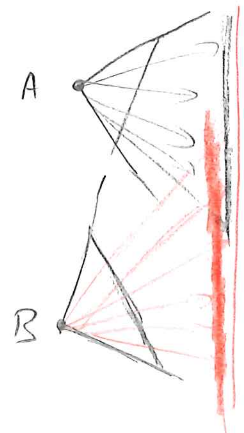
Eqn 2: $\vec{P}_2 = \underbrace{KRK^{-1}}_{\text{3x3 matrix: homography!}}\vec{P}_1$

Stereo Rectification

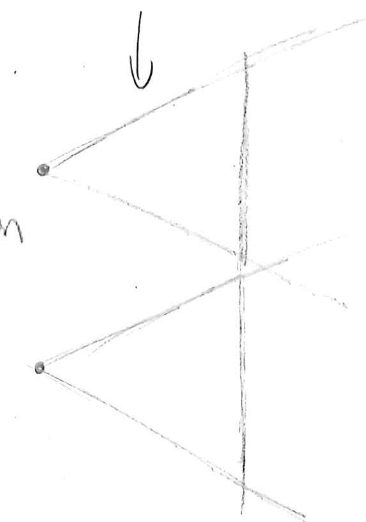
What we want:

Same orientation
Same F
X translation only.

What we get:

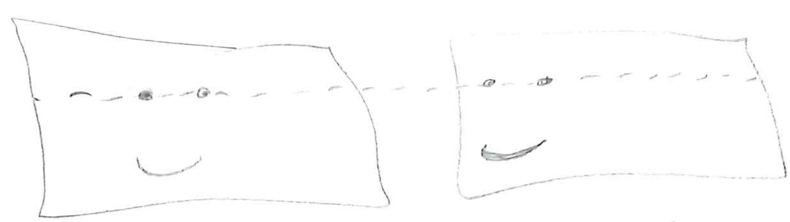


1. Let A be at origin (WLOG).
2. Project images onto a common plane (just a homograph!) simply rotating each camera.



Stereo pairs can be <u>rectified</u> by mapping their images onto a common plane using a homography for each image. Geometrically, this is simply applying a rotation to each camera (and possibly adjusting for differing intrinsics.

Once rectified, our stereo pairs look friendly:



We can search along rows for matching windows to find disparity and compute depth, because the only translation is in X.

Projective Geometry: Homogeneous lines

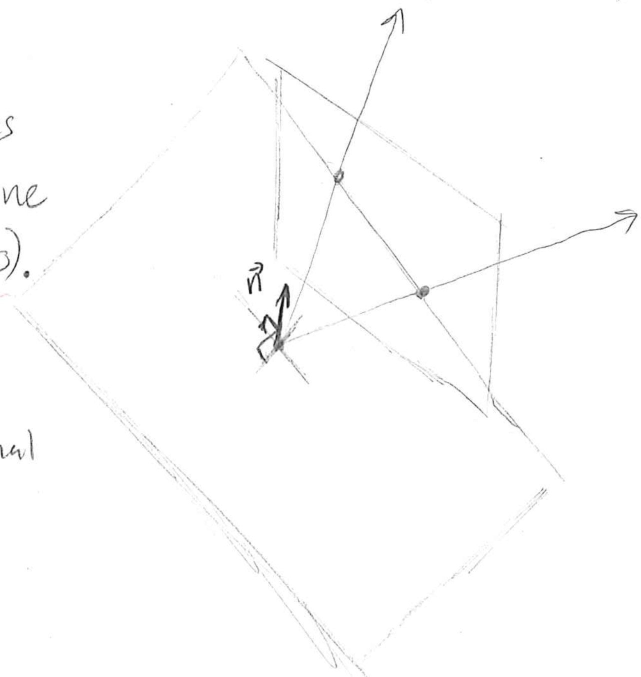
A point in $\mathbb{P}^2$ is (like) a ray in 3D, projected onto a plane.

Can we represent lines in $\mathbb{P}^2$? Yes! (we can represent conics, etc. too! out of scope in this class)

A point is 0D, we represent it as a 1D thing (ray).

A line is 1D, we represent it as a 2D thing (plane!)

A line in $\mathbb{P}^2$ can be interpreted as the set of (homog.) points that lie on a plane in $\mathbb{R}^3$ that passes through (0,0,0).



We represent the (line) plane using its normal vector: the vector orthogonal to the plane at the origin.

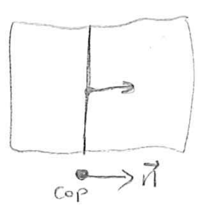
$$\vec{n} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

In 2D, this projects to the line $ax + by + c = 0$.

Examples:

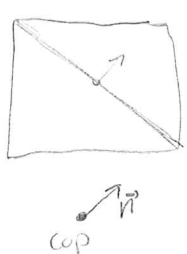
$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$ $1x + 0y + 0 = 0$

Line: $x = 0$ (vertical)



$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$ $x + y = 0$

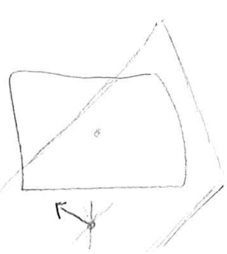
$y = -x$



$y = 2x + 4$

$-2x + y - 4 = 0$

$[-2 \;\; 1 \;\; -4]$



Notice:

Lines have a scale ambiguity just like points do:

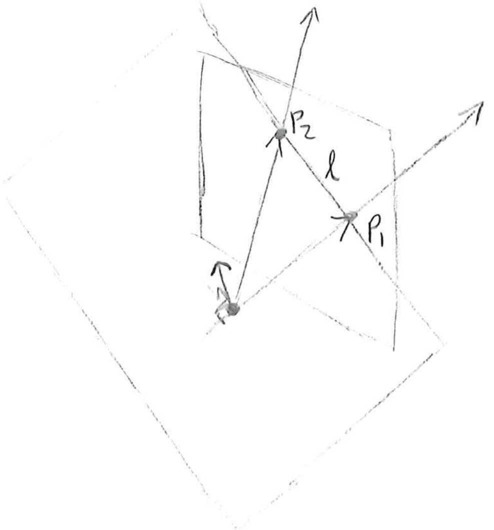
$$kax + kby + kc = 0$$

is the same line as

$$ax + by + c = 0$$

for any $k \neq 0$.

Projective Geometry: Point-Line duality



The line through two points (2D) is the plane spanned by their two 3-vectors (3D).

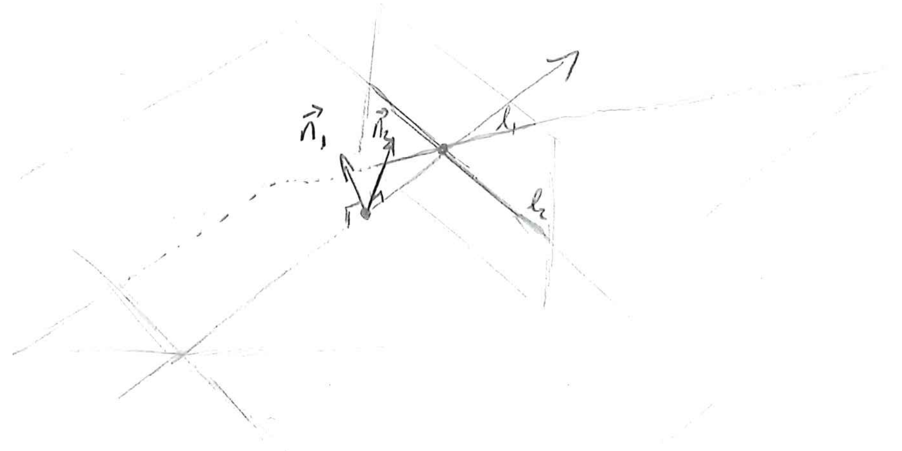
The plane normal vector is the vector orthogonal to both points' vectors:

cross product!

$$\ell = P_1 \times P_2$$

The point of intersection of two lines (2D) is the vector that lies on both planes. Such a vector is orthogonal to both plane normals!

$$p = \ell_1 \times \ell_2$$



Computing Cross Products

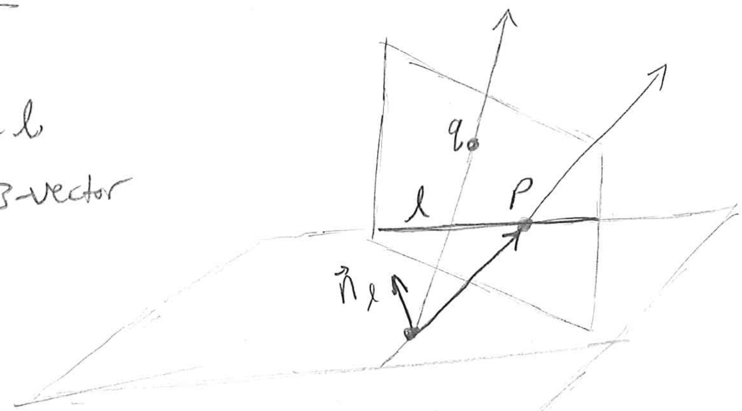
$$\begin{array}{c}P_1\\ \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix}\end{array} \times \begin{array}{c}P_2\\ \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix}\end{array} = \begin{bmatrix} y_1 z_2 - z_1 y_2 \\ z_1 x_2 - x_1 z_2 \\ x_1 y_2 - y_1 x_2 \end{bmatrix} \quad \text{(yuck!)}$$

Fact: this can be written as a matrix multiplication:

$$\begin{bmatrix} 0 & -z_1 & y_1 \\ z_1 & 0 & -x_1 \\ -y_1 & x_1 & 0 \end{bmatrix} \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix} = [P_1]_\times \cdot P_2$$

So $[P]_\times$ means form the 3x3 cross product matrix for P, so we can compute it using a matrix multiply (= dot product).

Projective Geometry: Points on lines, lines through points



Geometrically

If point p is on line l
Then p's homogeneous 3-vector
lies on l's 3D plane.

Consequence:

$$p \cdot l = 0$$

if and only if p lies on l.

Similarly (equivalently!), a line l goes through a point p iff $p \cdot l = 0$.

Algebraically

$l = [a \ b \ c]^T$ represents 2D line $ax + by + c = 0$

$p = [x \ y \ z]^T$ represents 2D point $\left(\frac{x}{z}, \frac{y}{z}\right) = (\hat{x}, \hat{y})$

p is on l if $a\hat{x} + b\hat{y} + c = 0$!

$$a\frac{x}{z} + b\frac{y}{z} + c = 0$$ multiply through by z

$$ax + by + cz = 0$$

$$[a \ b \ c]\begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

$$l \cdot p = 0$$