Lecture 12: Transformations
2D Linear and Affine Transformations
Announcements

- To use slip days, send me email after you've submitted late.
Goals

• Know what is possible with 2D linear transformations: (scale, shear, rotation)

• Understand the motivation and math behind homogeneous coordinates.

• Know what is possible with 2D affine transformations: (all of the above, plus translation)
Running motivational example: Panorama Stitching
Where are we?

1. How do we describe the transformation?
2. How do we find an accurate transformation?
3. How do we actually warp the image?

- linear, affine, or projective transformations
- least squares
- RANSAC
- inverse warping
Parametric (global) Warping

- Apply the same function to all coordinates.

\[ p = (x, y) \quad T \text{ transforms image coordinates } p' = (x', y') \]

\[ x', y' = T(x, y) \]

Self-imposed restriction: \( T \) is a matrix.

\[
\begin{bmatrix}
    x' \\
    y'
\end{bmatrix}
= \begin{bmatrix}
    t_{11} & t_{12} \\
    t_{21} & t_{22}
\end{bmatrix}
\begin{bmatrix}
    x \\
    y
\end{bmatrix}
\]

What can we do with this?
2x2 Matrices

\[
\begin{bmatrix}
x' \\
y'
\end{bmatrix} = \begin{bmatrix} t_{11} & t_{12} \\ t_{21} & t_{22} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}
\]

A 2x2 matrix can represent all possible linear transformations on input coordinates. (each output coordinate = linear function of input coordinates)
Scale

Input

Uniform Scale

Nonuniform Scale

\[ x' = 2x \]
\[ y' = 2y \]

\[ \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \]

leave y unchanged
Shear

Input

Shear (x)

Shear (both)

\[
\begin{align*}
X' &= x + 0.5y \\
y' &= y
\end{align*}
\]
Reflection

Input

Reflection (across y)

Reflection (across x)
Rotation

Input

Rotation about the origin

\[
\begin{pmatrix}
\cos \theta & -\sin \theta \\
\sin \theta & \cos \theta
\end{pmatrix}
\]
Translation

Input

Translation by \((u, v)\)
Today's Problems

1. What's the fewest correspondences you could use to (unambiguously) find a 2x2 linear transformation $T$?

2. Come up with a matrix that represents a translation.
Matrices can't translate.

We'll use a clever math hack to make them do it anyway:

Homogeneous Coordinates

Use a 3D vector to represent a 2D point. Always put a 1 in the third dimension.
Matrices can't translate.

We'll use a clever math hack to make them do it anyway:

**Homogeneous Coordinates**

Use a \(3D\) vector to represent a \(2D\) point.
Always put a \(1\) in the third dimension.
Matrices can't translate.

We'll use a clever math hack to make them do it anyway:

**Homogeneous Coordinates**

Use a **3D** vector to represent a **2D** point.
Always put a **1** in the third dimension.
How do we transform these?

\[
\begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
t_{11} & t_{12} & t_{13} \\
t_{21} & t_{22} & t_{23} \\
0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
x \\
y \\
1
\end{pmatrix}
= 
\begin{pmatrix}
x' \\
y' \\
1
\end{pmatrix}
\begin{pmatrix}
t_{11}x + t_{12}y + t_{13} \\
t_{21}x + t_{22}y + t_{23} \\
0
\end{pmatrix}
\]
Interactive Demo

- https://iis.uibk.ac.at/public/piater/courses/demos/homography/homography.xhtml
Affine Transformations

The transformations possible with a 3x3 matrix like this

\[
\begin{bmatrix}
t_{11} & t_{12} & t_{13} \\
t_{21} & t_{22} & t_{23} \\
0 & 0 & 1
\end{bmatrix}
\]

are called affine transformations.

Affine basically means linear plus shift:

- \( f(x, y) = ax + by \) is linear
- \( f(x, y) = ax + by + c \) is affine

each input coordinate is an affine function of the input coordinates
Which of these can be done by a 2D linear transformation?

\[ y' = ax + by \]
Which of these can't be done by a 2D affine transformation?
Transformations: Properties

• Anything you get from matrix multiplication comes for free!

  • Associative! Composable!
  • Not commutative.
Linear Transformations: Properties

- Linear transformations
  \[
  \begin{bmatrix}
  x' \\
  y' \\
  w
  \end{bmatrix}
  =
  \begin{bmatrix}
  a & b & 0 \\
  d & e & 0 \\
  0 & 0 & 1
  \end{bmatrix}
  \begin{bmatrix}
  x \\
  y \\
  w
  \end{bmatrix}
  \]

- Properties:
  - Lines map to lines
  - Parallel lines remain parallel
  - Ratios of lengths along lines are preserved
  - Closed under composition
  - Origin maps to origin
Affine Transformations: Properties

- Affine transformations

\[
\begin{bmatrix}
    x' \\
y' \\
w
\end{bmatrix} = \begin{bmatrix}
    a & b & c \\
d & e & f \\
0 & 0 & 1
\end{bmatrix} \begin{bmatrix}
    x \\
y \\
w
\end{bmatrix}
\]

- Properties:
  - Lines map to lines
  - Parallel lines remain parallel
  - Ratios of lengths along lines are preserved
  - Closed under composition
  - Origin does not necessarily map to origin
We've found correspondence. What's the transformation?

\[
\begin{bmatrix}
x' \\
y' \\
1
\end{bmatrix} = \begin{bmatrix}
x \\
y \\
1
\end{bmatrix}
\]
We've found the transformation. How do we warp the image?

\[
\begin{bmatrix}
  x' \\
  y' \\
  1
\end{bmatrix} =
\begin{bmatrix}
  1 & 0 & -115 \\
  0 & 1 & -3 \\
  0 & 9 & 1
\end{bmatrix}
\begin{bmatrix}
  x \\
  y \\
  1
\end{bmatrix}
\]
Warping

x -> x'
y -> y'

mountain landscape before and after warping
Warping
Forward Warping

$\mathbf{T}(x, y)$
We've found the transformation. How do we warp the image?

\[
\begin{bmatrix}
x' \\
y' \\
1
\end{bmatrix} = \begin{bmatrix}
1 & 0 & -114.8 \\
0 & 1 & -2.9 \\
0 & 9 & 1
\end{bmatrix} \begin{bmatrix}
x \\
y \\
1
\end{bmatrix}
\]
Inverse Warping

\[ T^{-1}(x, y) \]
Bilinear Interpolation