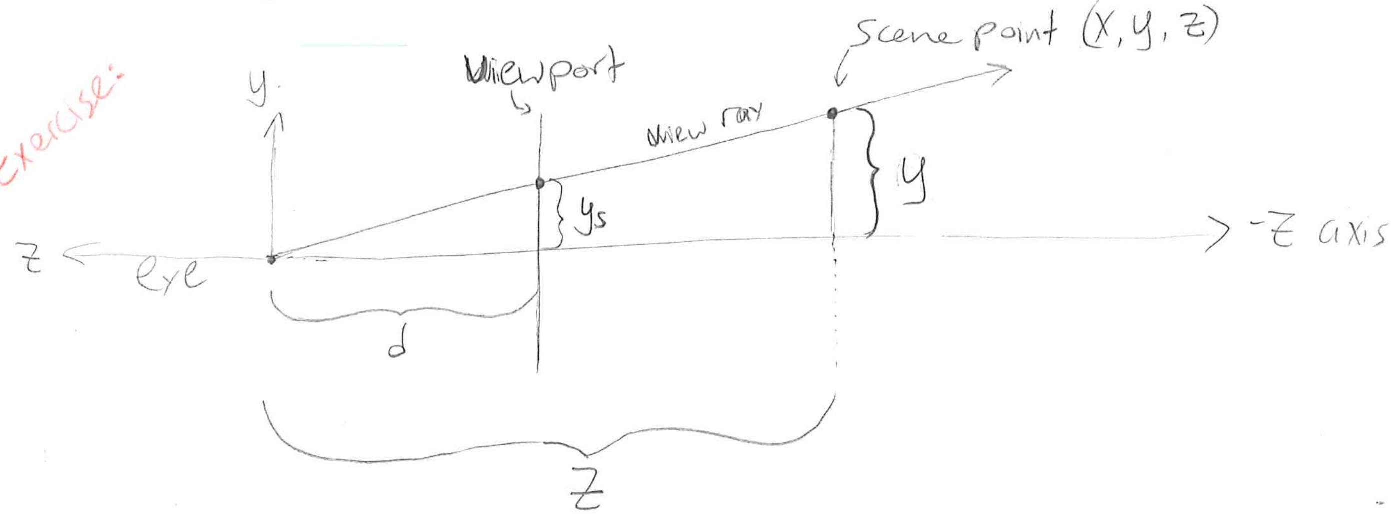
What about Perspective Cameras?

(down $-x$)
Looking side-on at a canonical perspective camera:

Exercise:



What is y_s in terms of (x, y, z, d)?

Similar triangles!?
$$\frac{y_s}{d} = \frac{y}{z}, \text{ or } \boxed{y_s = \frac{d}{z} y}$$

We'd like a matrix that does this:

$$\begin{bmatrix} \frac{dx}{z} \\ \frac{dy}{z} \\ ?\,z \\ 1 \end{bmatrix} = \begin{bmatrix} \quad & \quad & \quad & \quad \\ & & & \\ & & & \\ & & & \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

Ideally, we'd have z alone; but this isn't easy - more later

But no such matrix exists! :(
Can't divide by z.

Projective Transformations

As they are, matrices can't tell us where a 3D point will land on the viewport of a perspective camera.

To fix this, we're going to extend our math hack (homogeneous coordinates) to allow w values $\neq 1$.

$$\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} \overset{\text{equivalent to}}{\sim} \begin{bmatrix} x/w \\ y/w \\ z/w \\ \underset{=w/w}{1} \end{bmatrix}$$

A homogeneous point is considered <u>equivalent</u> to all possible scalings of itself.

$\vec{x} \sim k\vec{x}$ for any k (assume $k \neq 0$)

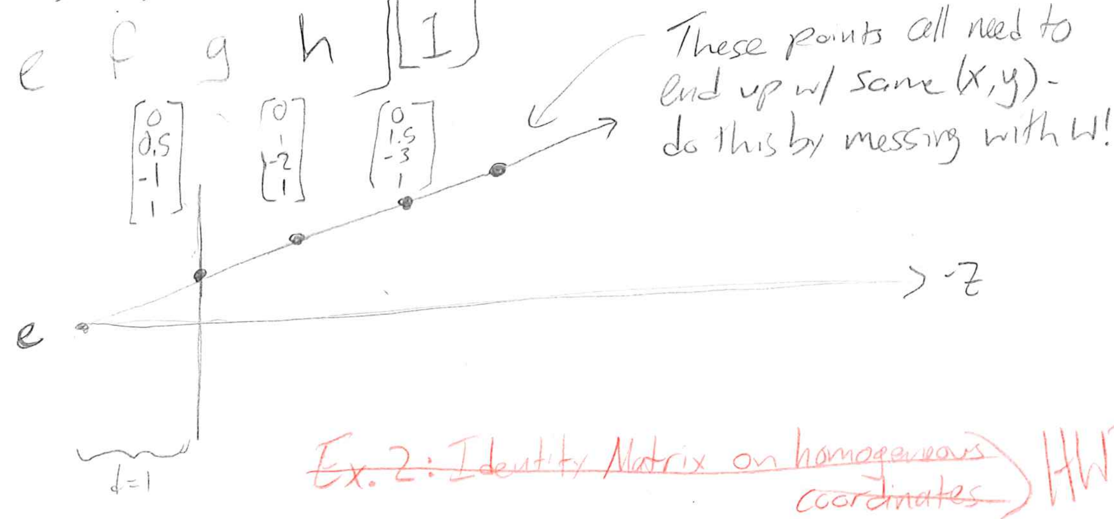
If you want to know what happens with $w=0$, you'll have to take Computer Vision!

Ex. 1: Homog. Point Equivalence

Now, a transformation matrix is free to mess with w!

$$\begin{bmatrix} \hat{x} \\ \hat{y} \\ \hat{z} \\ \hat{w} \end{bmatrix} = \begin{bmatrix} a_1x+b_1y+c_1z+d_1 \\ a_2x+b_2y+c_2z+d_2 \\ a_3x+b_3y+c_3z+d_3 \\ ex+fy+gz+h \end{bmatrix} \begin{bmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \\ e & f & g & h \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

$$x' = \hat{x}/\hat{w}$$
$$y' = \hat{y}/\hat{w}$$
$$z' = \hat{z}/\hat{w}$$



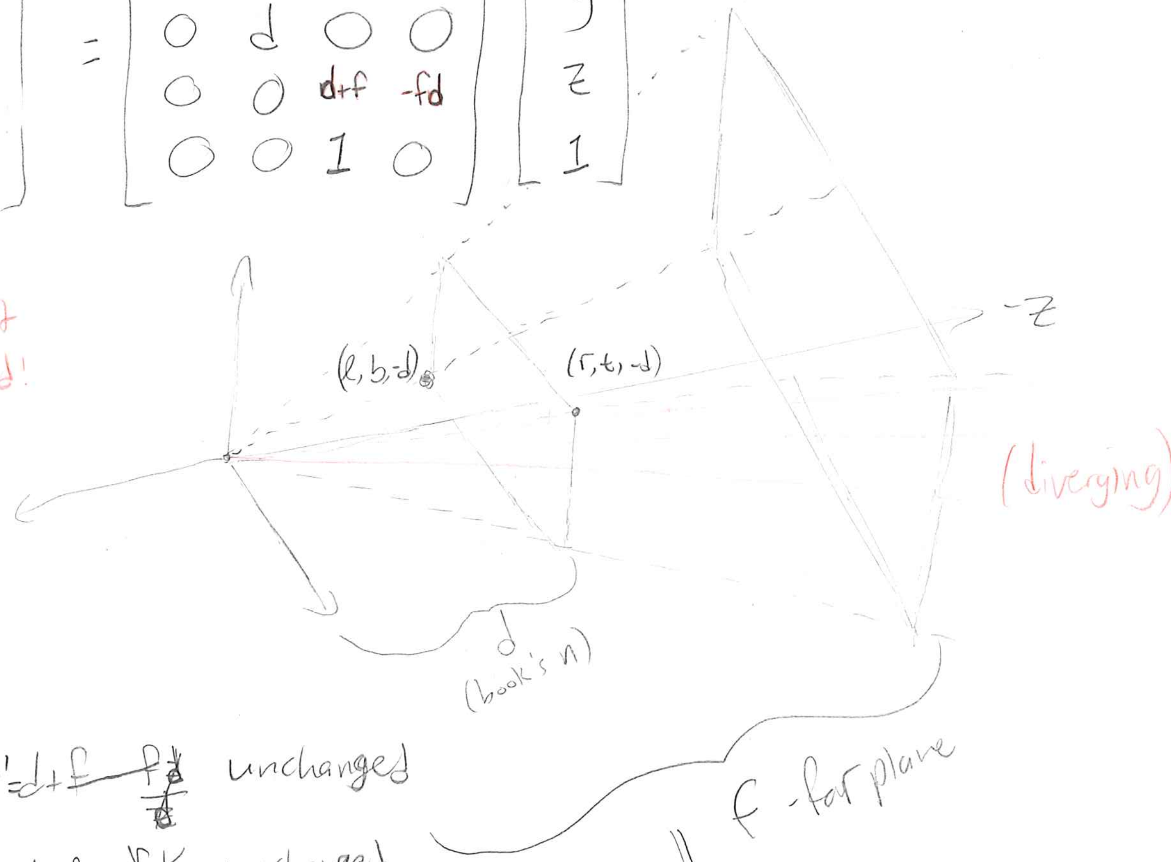
~~Ex. 2: Identity Matrix on homogeneous coordinates~~ HW2

Demo: Projective Transforms

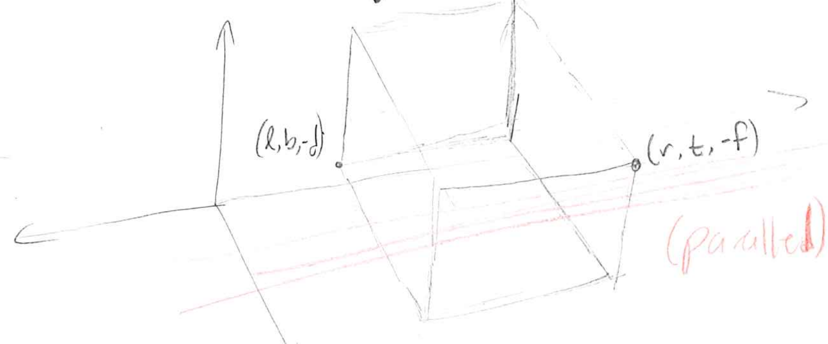
Recall, we wanted:

$$\begin{bmatrix} dx \\ dy \\ \boxed{(d+f)z-fd} \\ z \end{bmatrix} \sim \begin{bmatrix} \frac{d}{z}x \\ \frac{d}{z}y \\ \boxed{d+f-\frac{fd}{z}} \\ 1 \end{bmatrix} = \overbrace{\begin{bmatrix} d & 0 & 0 & 0 \\ 0 & d & 0 & 0 \\ 0 & 0 & d+f & -fd \\ 0 & 0 & 1 & 0 \end{bmatrix}}^{M_{persp}} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

this will get divided by z, so we can't just keep z unchanged!



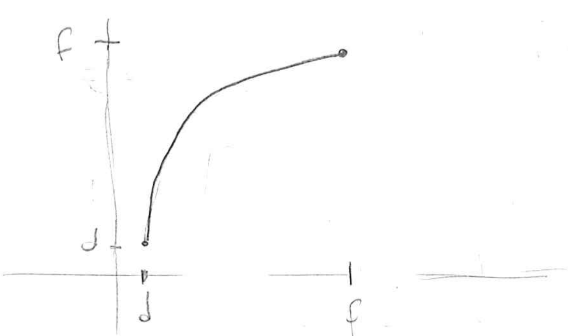
M_{persp}



$$z' = d+f-\frac{fd}{z}$$

If $z=d$, $z' = d + \cancel{f} - \cancel{\frac{fd}{d}}$ unchanged

$z=f$, $z' = \cancel{d} + f - \cancel{\frac{fd}{f}}$ unchanged



At this point, the view volume looks just like it did before orthographic projection! Rays parallel, but wrong dimensions.

So re-use the orth matrix: $\boxed{M_{proj} = M_{orth} M_{persp}}$