Viewing Transformations: Overview

$$\begin{bmatrix} x_{pixel} \\ y_{pixel} \\ z_{pixel} \\ 1 \end{bmatrix} = \overset{\text{(viewport matrix)}}{M_{vp}} \; \overset{\text{(projection matrix)}}{M_{proj}} \; \overset{\text{(camera matrix)}}{M_{cam}} \; \overset{\text{(model matrix)}}{M_{model}} \begin{bmatrix} x_{model} \\ y_{model} \\ z_{model} \\ 1 \end{bmatrix}$$

image space

normalized device / Canonical view volume

eye

world

model coords

Wireframe Rendering Algorithm

Renders line segments as its primitives (not triangles)
(↳ can extend to triangles later)

Input: A set of line segments $\{(a_i, b_i) : 0 \le i < n\}$

1. Form all matrices (M_{vp}, M_{proj}, M_{cam}, M_{model})

2. $M \leftarrow (M_{vp} \cdot M_{proj} \cdot M_{cam} \cdot M_{model})$

3. for each line segment $\vec{a}_i, \vec{b}_i$:

    $p = M\vec{a}_i$

    $q = M\vec{b}_i$

    draw_line$((x_p, y_p), (x_q, y_q))$

Viewport Matrix

- Suppose we modeled our scene in the cube $[-1,1]^3$

Input: scene in Canonical View Volume
normalized device coordinates

- all visible points in a cube of side length 2 centered at the origin: $(x, y, z) \in [-1, 1]^3$

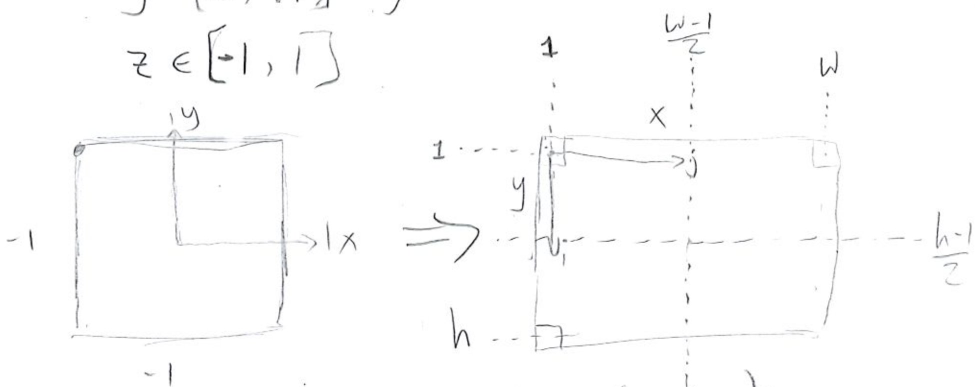
Parameters: W, H - image dimensions

Output: all visible points in pixel coordinates

$x \in [0.5, W+0.5]$

$y \in [0.5, H+0.5]$

$z \in [-1, 1]$



(Julia (conventions)):

$$x_p = \left(\frac{x_v+1}{2}\right)w + \frac{1}{2}$$

$$y_p = \left(\frac{-y_v+1}{2}\right)h + \frac{1}{2}$$

$$z_p = z_v$$

Scale x 2 -> width

y 2 -> height

plug in zero

Translate $(0,0) \to \left(\frac{w+1}{2}, \frac{h+1}{2}\right)$

Flip y axis; leave z unchanged

-> why? We'll use it later to know what is in front of what.

$\boxed{\text{Ex}}$ write a matrix that does this

$$\begin{pmatrix} x_p \\ y_p \\ z_p \\ 1 \end{pmatrix} = \begin{bmatrix} w/2 & 0 & 0 & \frac{w+1}{2} \\ 0 & -h/2 & 0 & \frac{h+1}{2} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} x_v \\ y_v \\ z_v \\ 1 \end{pmatrix}$$

$$x_p = \frac{w}{2}x_v + \frac{w}{2} + \frac{1}{2}$$

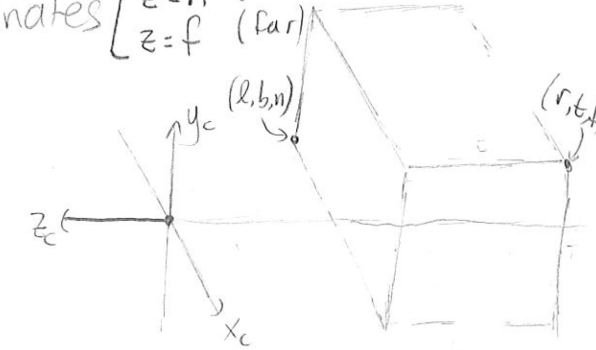
$$y_p = \frac{h}{2}y_v + \frac{h}{2} + \frac{1}{2}$$

Projection Matrix - Orthographic

Suppose we modeled our scene in a canonically posed orthographic camera's view area.

Input: Camera coordinates
Parameters: Orthographic viewport dimensions
Output: normalized device coordinates

$$\begin{cases} x = l & \text{(left)} \\ x = r & \text{(right)} \\ y = b & \text{(bottom)} \\ y = t & \text{(top)} \\ z = n & \text{(near)} \\ z = f & \text{(far)} \end{cases}$$



1. Translate $(l,b,n) \to (0,0,0)$

2. Scale $x: r-l \to 2$
   $y: t-b \to 2$
   $z: f-n \to 2$

3. Translate $(1,1,1)$ to $(0,0,0)$

$-l\left(\frac{2}{r-l}\right)$

$\frac{-2l}{r-l}$

Ex Write this matrix

$$\begin{bmatrix} I_{3\times3} & \begin{matrix} -1 \\ -1 \\ -1 \end{matrix} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{2}{r-l} & 0 & 0 & 0 \\ 0 & \frac{2}{t-b} & 0 & 0 \\ 0 & 0 & \frac{2}{f-n} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} I_{3\times3} & \begin{matrix} -l \\ -b \\ -n \end{matrix} \\ 0 & 1 \end{bmatrix}$$

$+\left(\frac{+2l}{r-l}\right) + \frac{r-l}{r-l} = \frac{r+l}{r-l}$

$$\begin{bmatrix} \frac{2}{r-l} & 0 & 0 & \frac{r+l}{r-l} \\ 0 & \frac{2}{t-b} & 0 & \frac{t+b}{t-b} \\ 0 & 0 & \frac{2}{f-n} & \frac{f+n}{f-n} \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} I_{3\times3} & \begin{matrix} -1 \\ -1 \\ -1 \end{matrix} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{2}{r-l} & 0 & 0 & \frac{-2l}{r-l} \\ 0 & \frac{2}{t-b} & 0 & \frac{-2b}{t-b} \\ 0 & 0 & \frac{2}{f-n} & \frac{-2n}{f-n} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Camera Matrix

Input: Scene in world coordinates

Parameters: camera frame $\vec{u}, \vec{v}, \vec{w}, \vec{e}$

Output: scene in canonically-positioned camera's coords
↳(eye at origin, looking down -z axis)

Ex Write this matrix

$$\begin{bmatrix} \vec{u} & \vec{v} & \vec{w} & \vec{e} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$ is the frame-to-canonical matrix!

Want: canonical-to-frame, so invert it! Algebraically, or

orthonormal basis, so $Q^T = Q^{-1}$

$$\begin{bmatrix} \boxed{\vec{u} \ \vec{v} \ \vec{w}} & \vec{e} \\ 0 \ 0 \ 0 & 1 \end{bmatrix}^{-1}$$ Let $\begin{bmatrix} \vec{u} & \vec{v} & \vec{w} \end{bmatrix}_{3\times3} = Q$

$$= \left(\begin{bmatrix} I_{3\times3} & \vec{e} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} Q & 0 \\ 0 & 1 \end{bmatrix} \right)^{-1}$$

$(AB)^{-1} = B^{-1}A^{-1}$

$$= \begin{bmatrix} Q^T & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} I_{3\times3} & -\vec{e} \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} Q^T & -Q^T\vec{e} \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \vec{u}^T & -\vec{u}^T\vec{e} \\ \vec{v}^T & -\vec{v}^T\vec{e} \\ \vec{w}^T & -\vec{w}^T\vec{e} \\ 0 & 1 \end{bmatrix}$$

Model Matrix: Whatever you need to put the object where you want it!