






Computer Graphics

Lecture 17
3D Transformations
Directions and Normals

Announcements

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  - Form project groups by Wednesday, 11/6 (two Wednesdays from now)

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  - Proposals due Friday 11/8
  - Take-home ~~mid~~lateterm exam out Friday 11/8, due start of class Tuesday 11/12

Goals

- Know how to apply affine transformations to direction vectors
- Know how to apply affine transformations to surface normals
- Know how form 3D affine transformation matrices for translation, scale, shear, and rotation about axes.

TODO Today

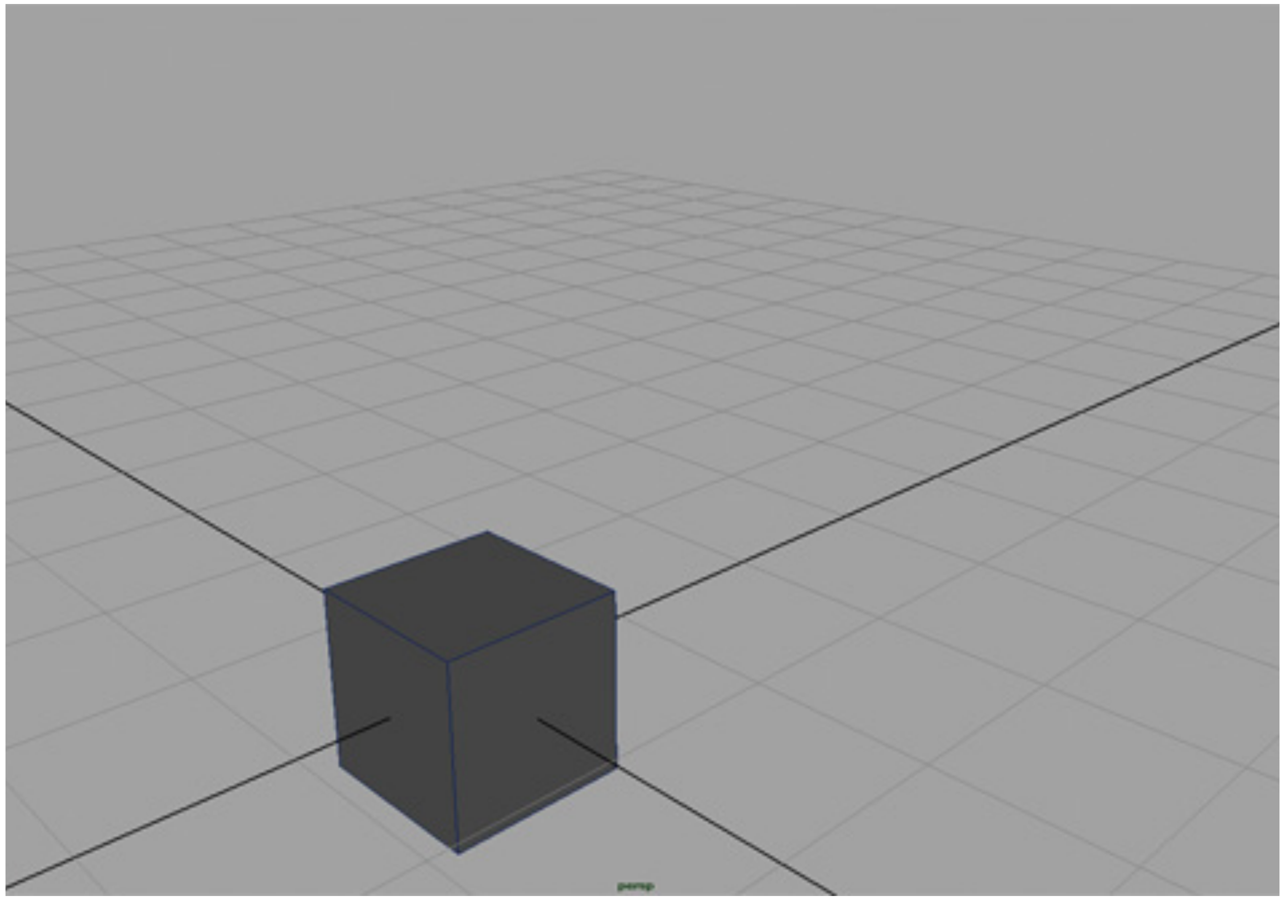
- 3D Affine Transformations
- 3D Rotations
- Transforming points vs vectors
- Transforming normals

Transformations in 3D

- Pretty much the same stuff
    - but with one additional D

Translation

$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & t_x \\ 0 & 1 & 0 & t_y \\ 0 & 0 & 1 & t_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$



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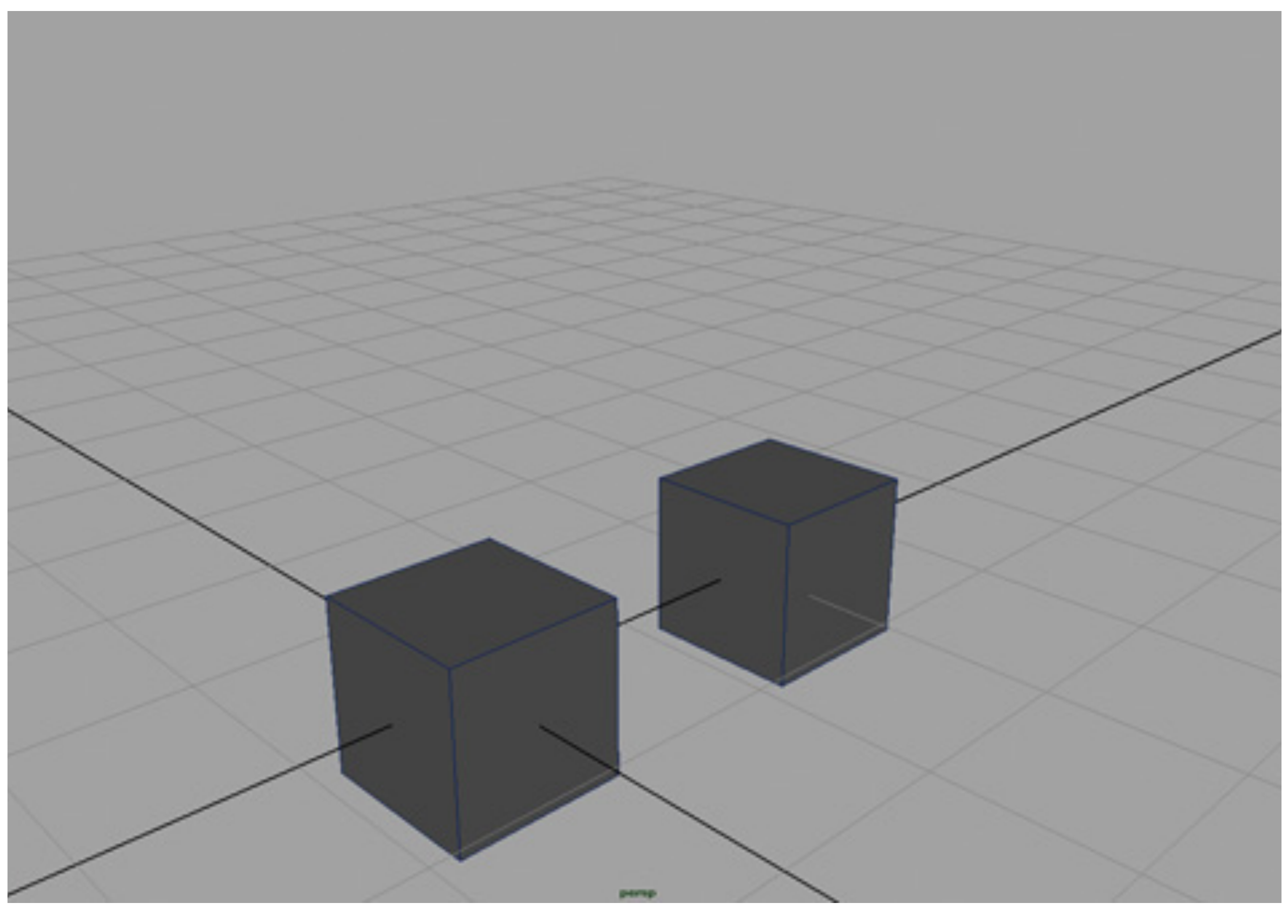

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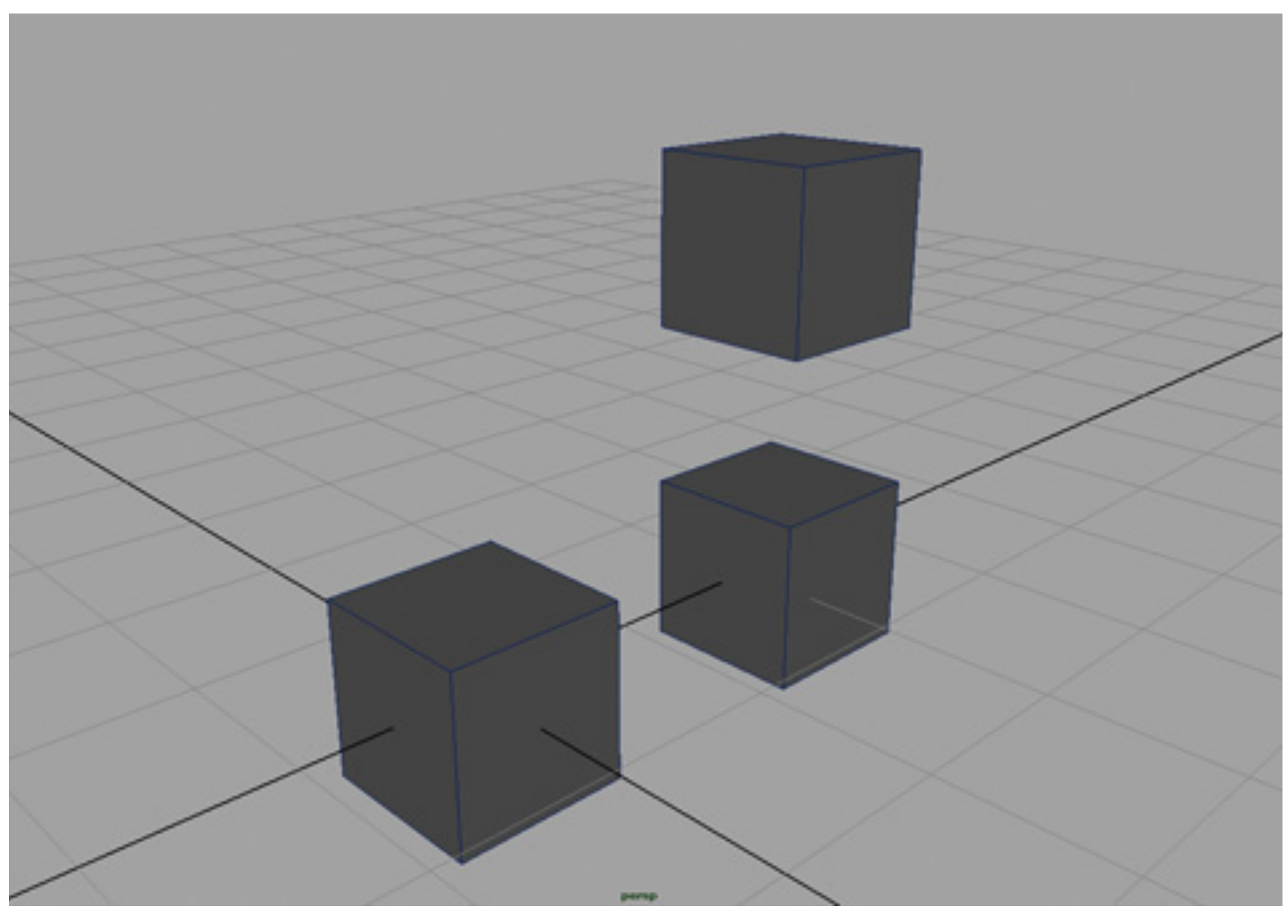

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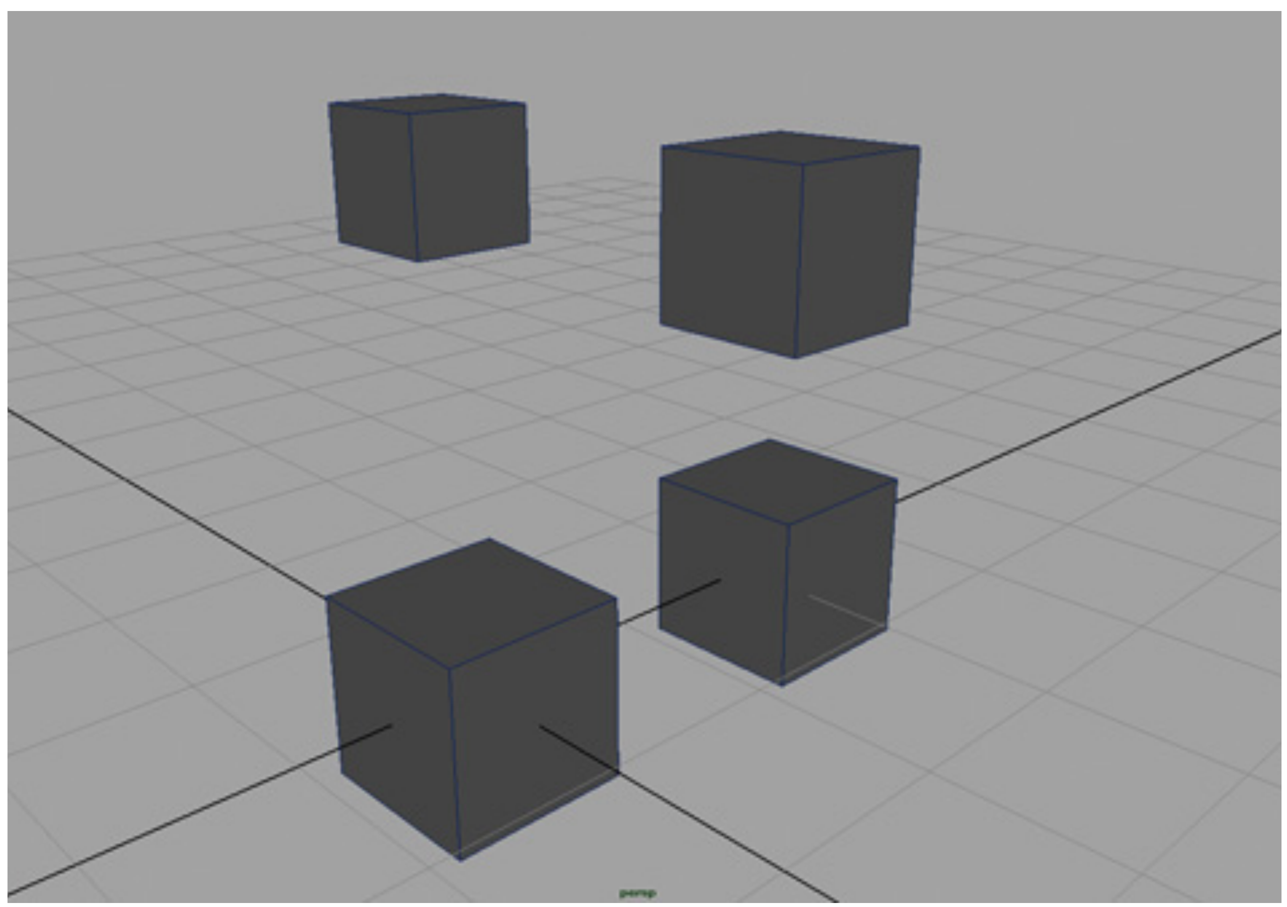
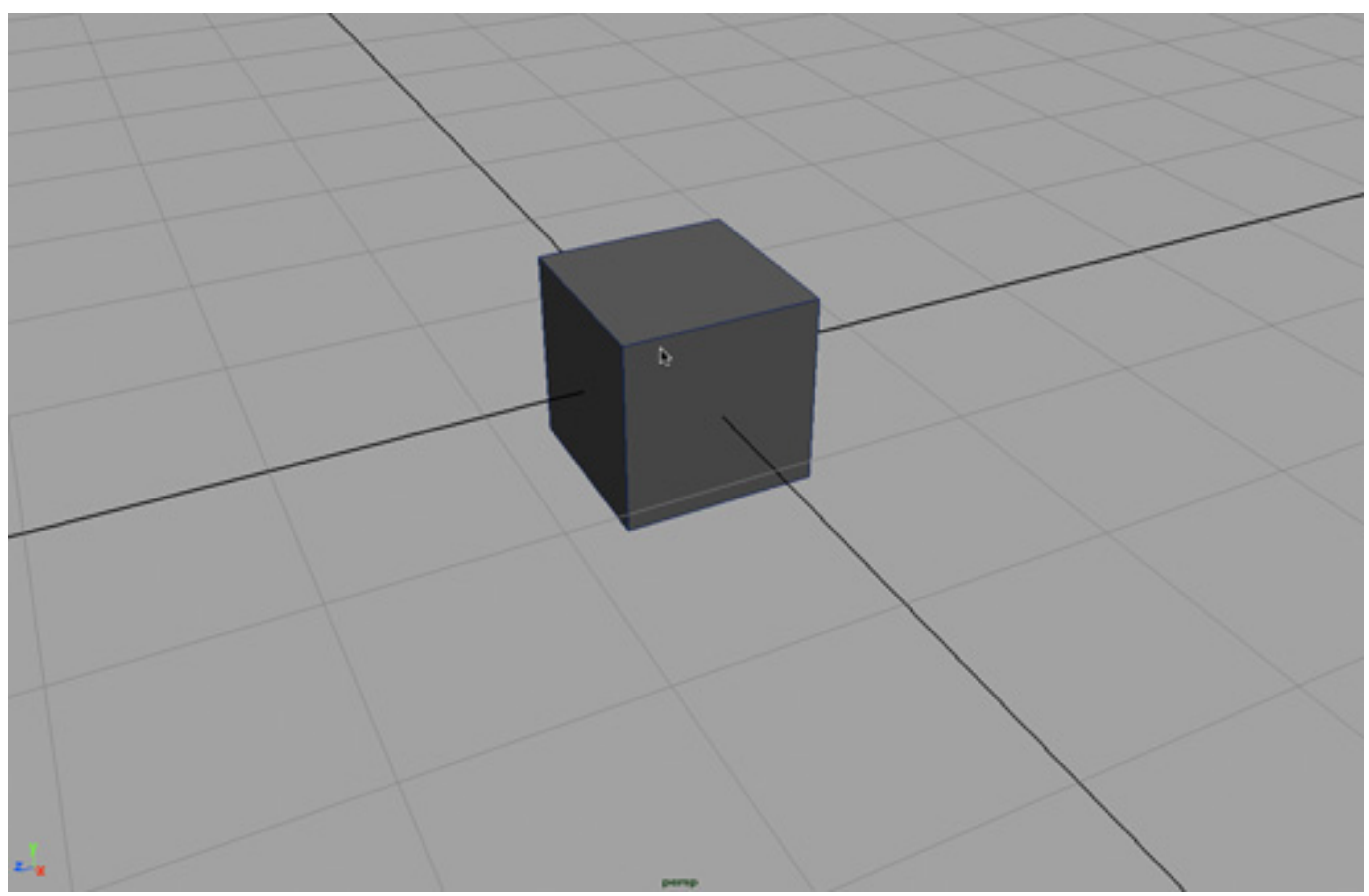


Scaling

$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} s_x & 0 & 0 & 0 \\ 0 & s_y & 0 & 0 \\ 0 & 0 & s_z & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$



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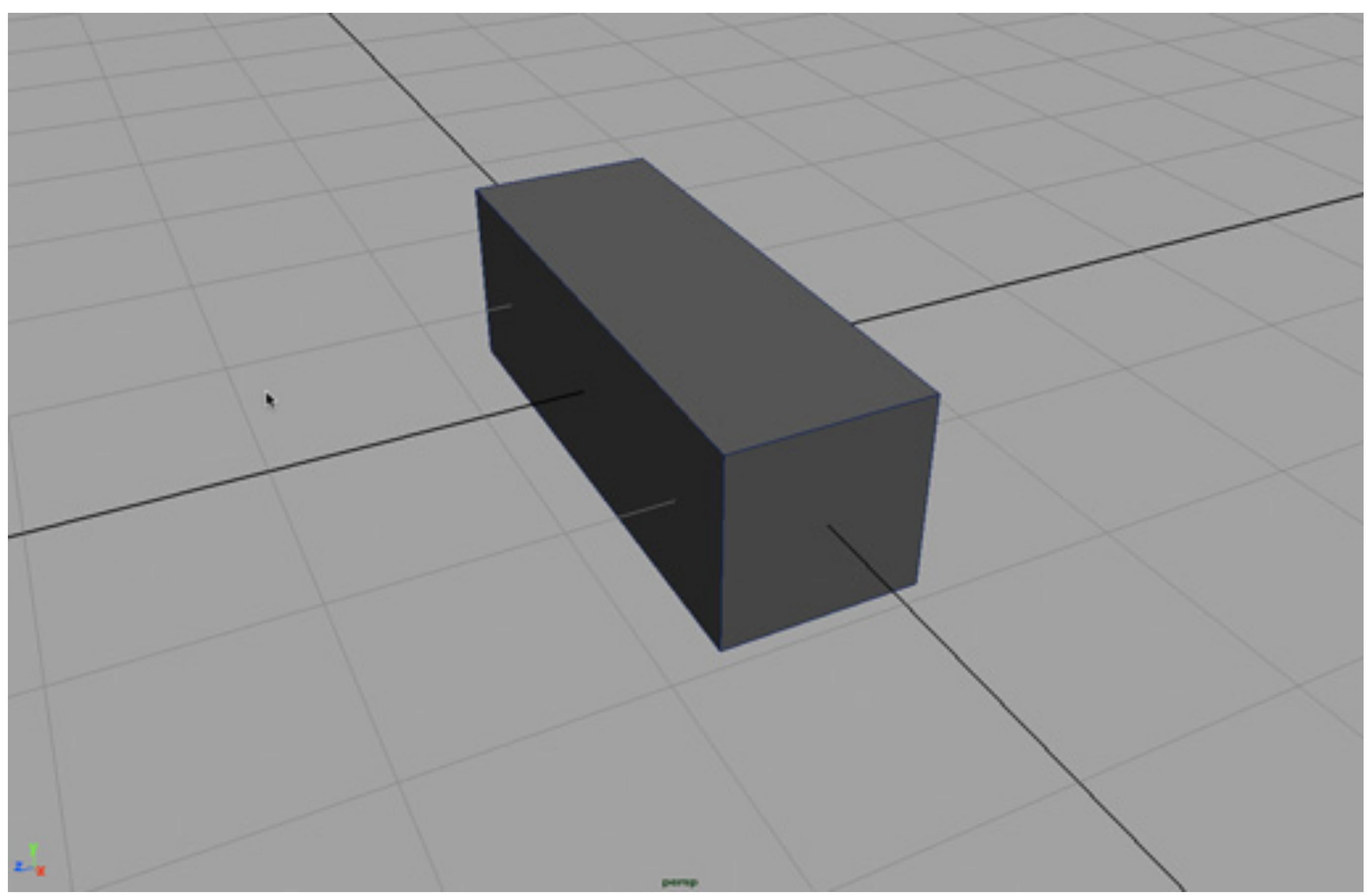

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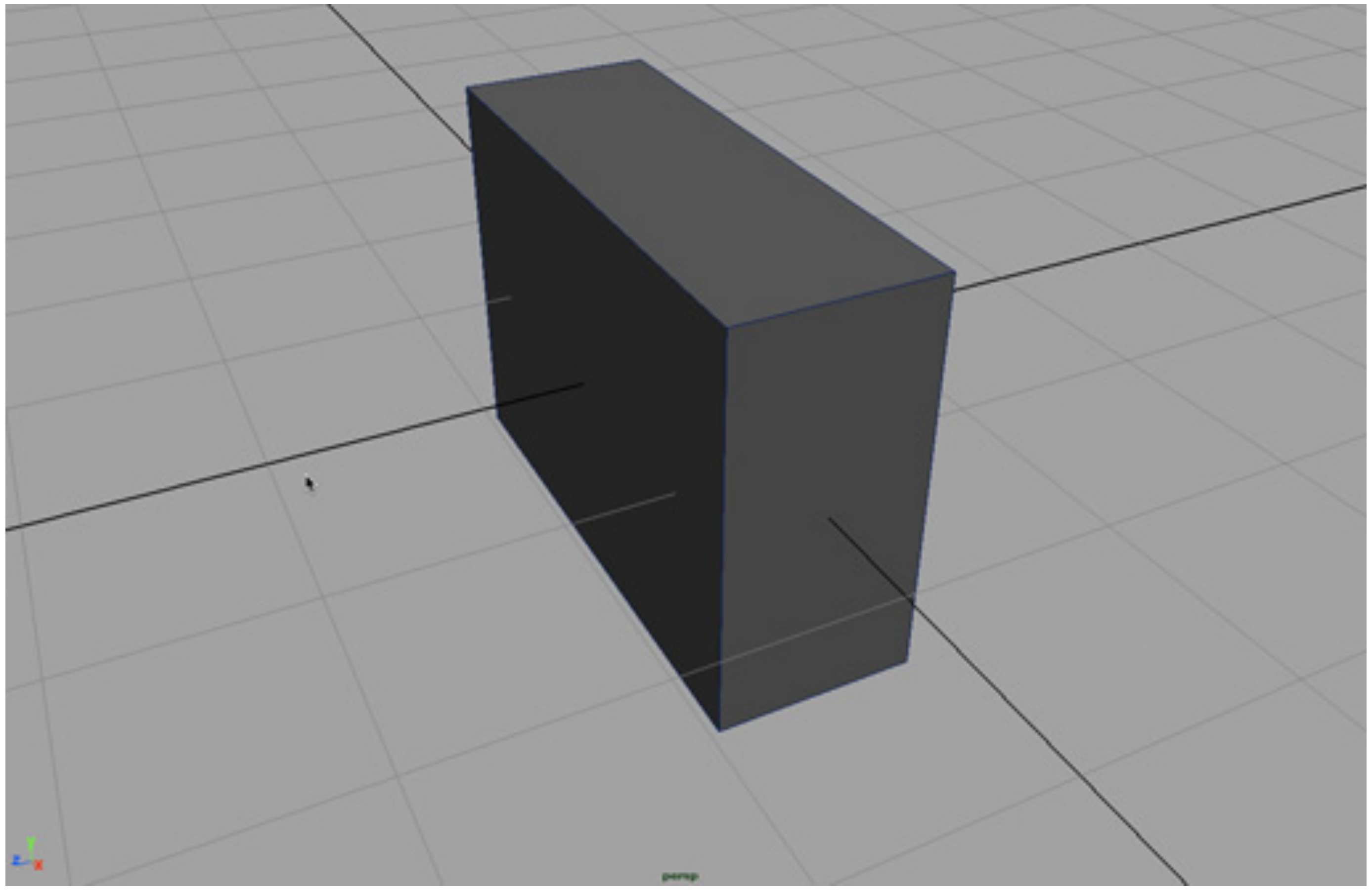

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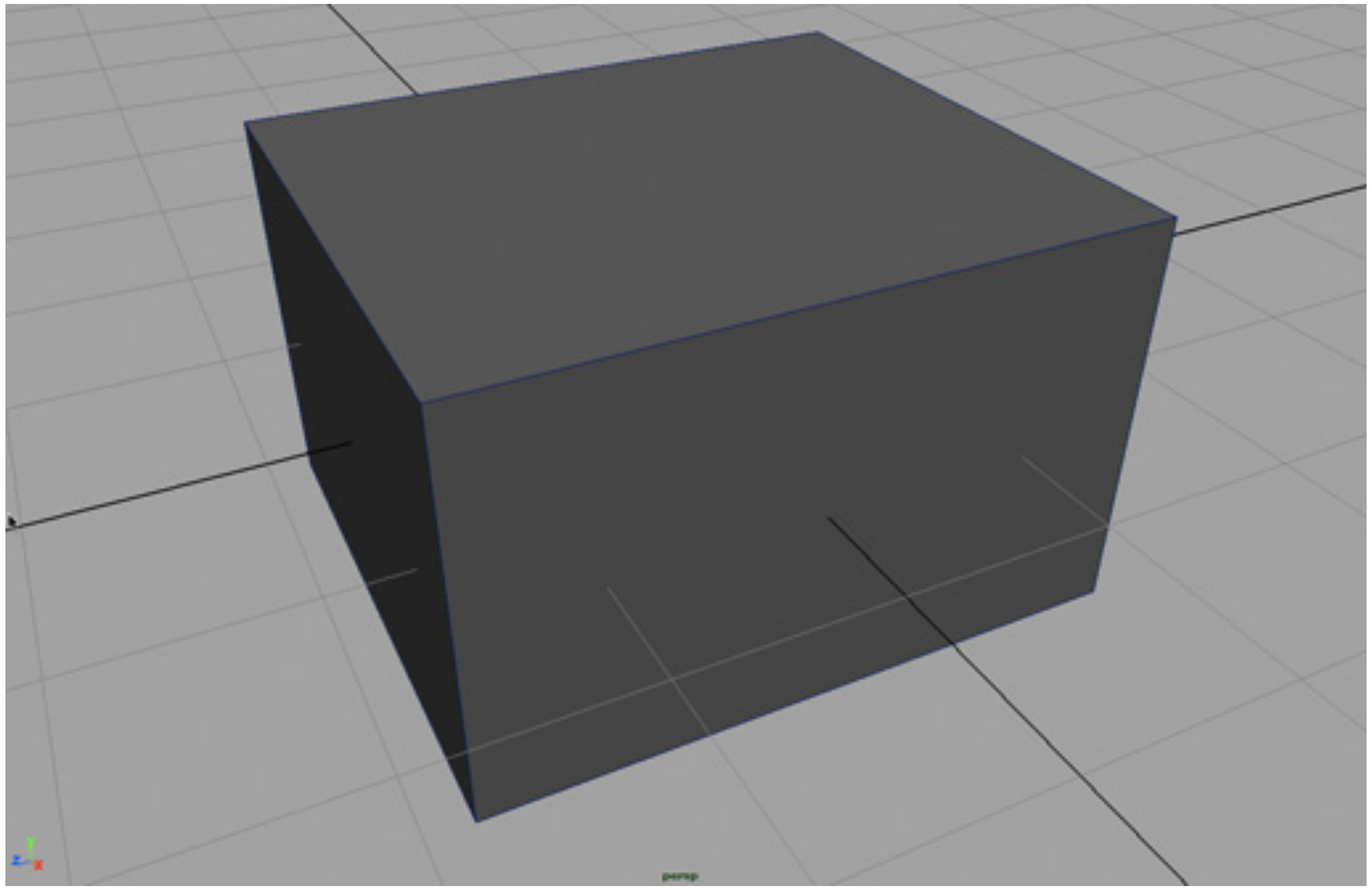

Rotations: A bit different

- A rotation in 2D is around a point
- A rotation in 3D is around an axis
  - so 3D rotation is w.r.t a line, not just a point
  - there are many more 3D rotations than 2D
    - a 3D space around a given point, not just 1D

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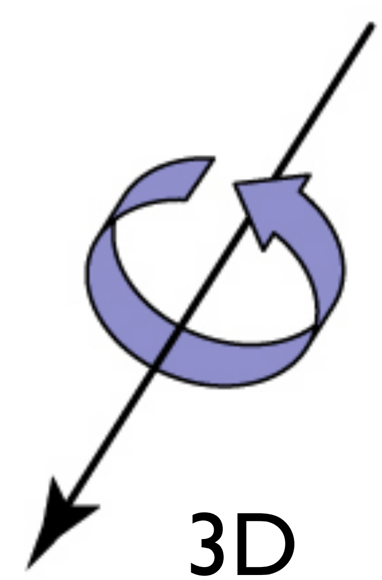
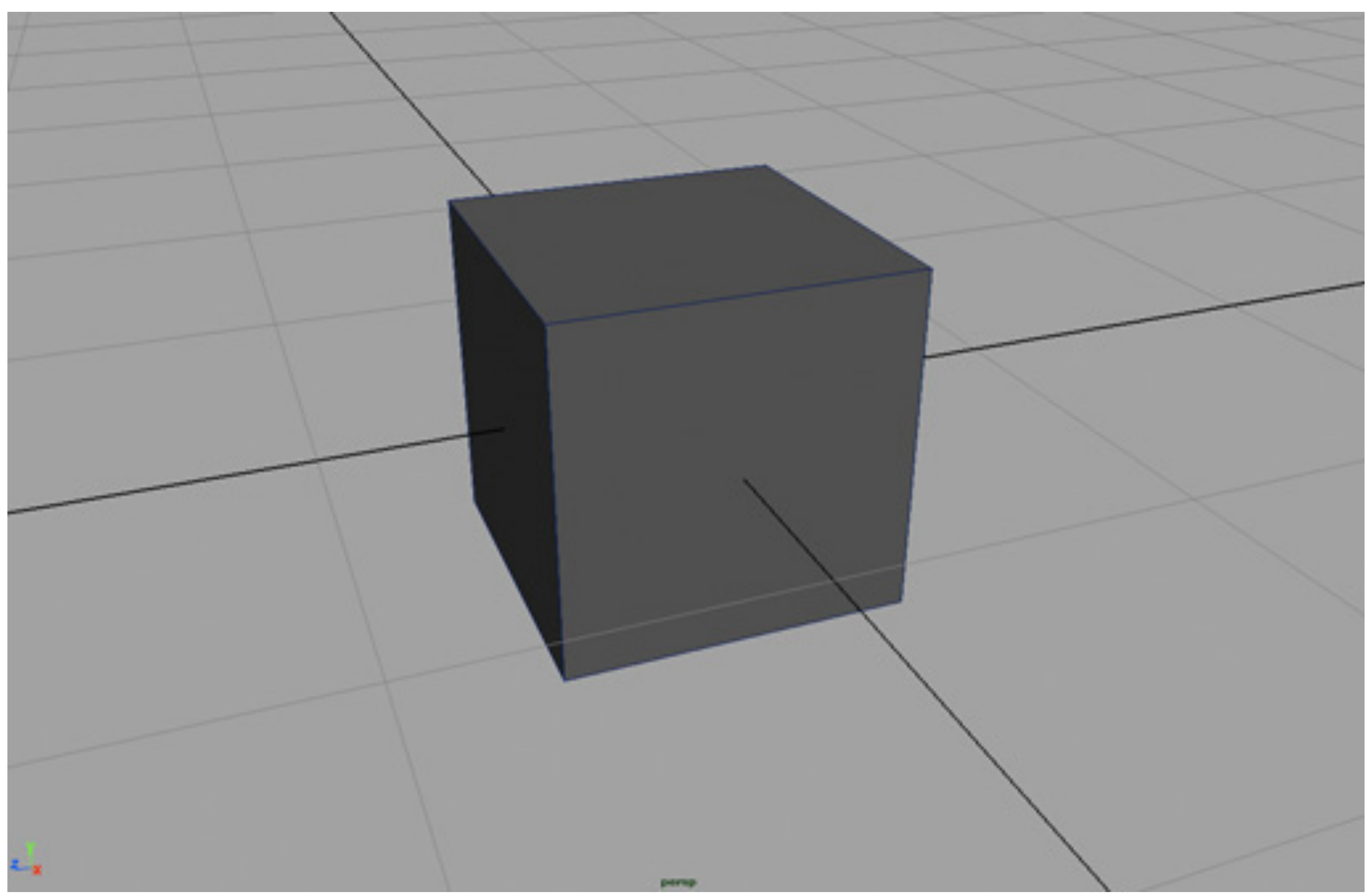
2D



3D

Rotation about z axis

$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$



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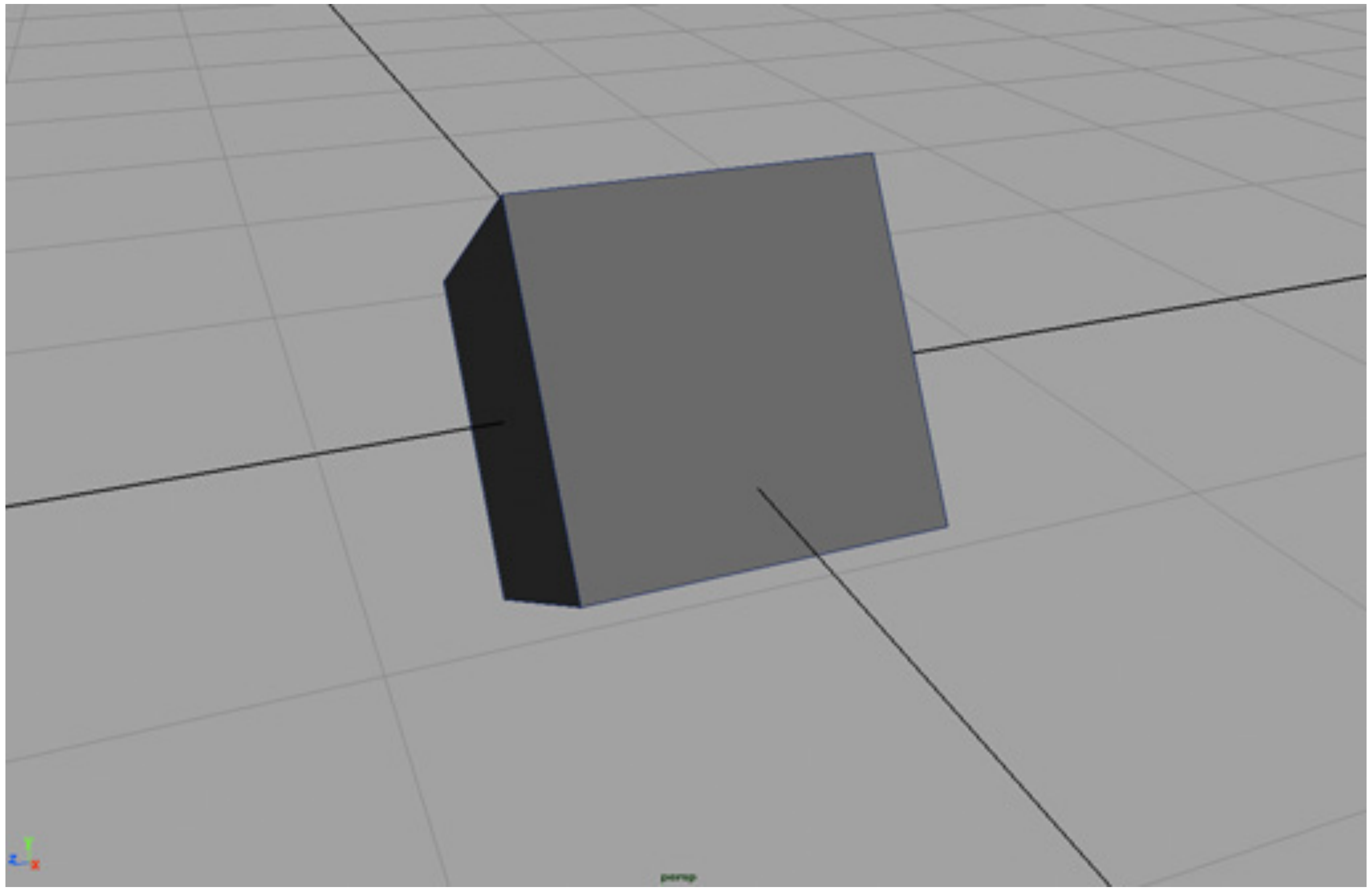
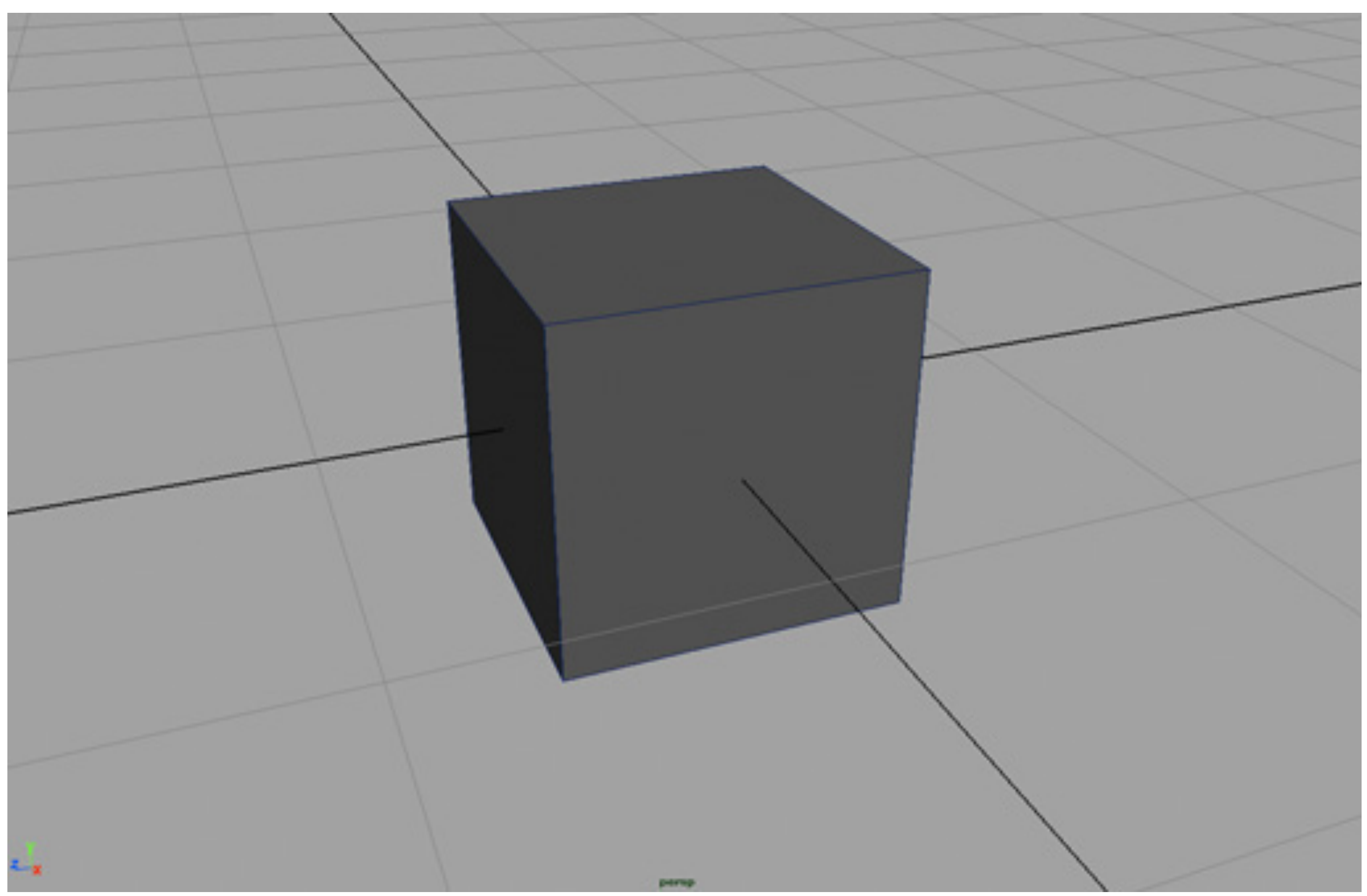


Rotation about *x* axis

$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta & 0 \\ 0 & \sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$



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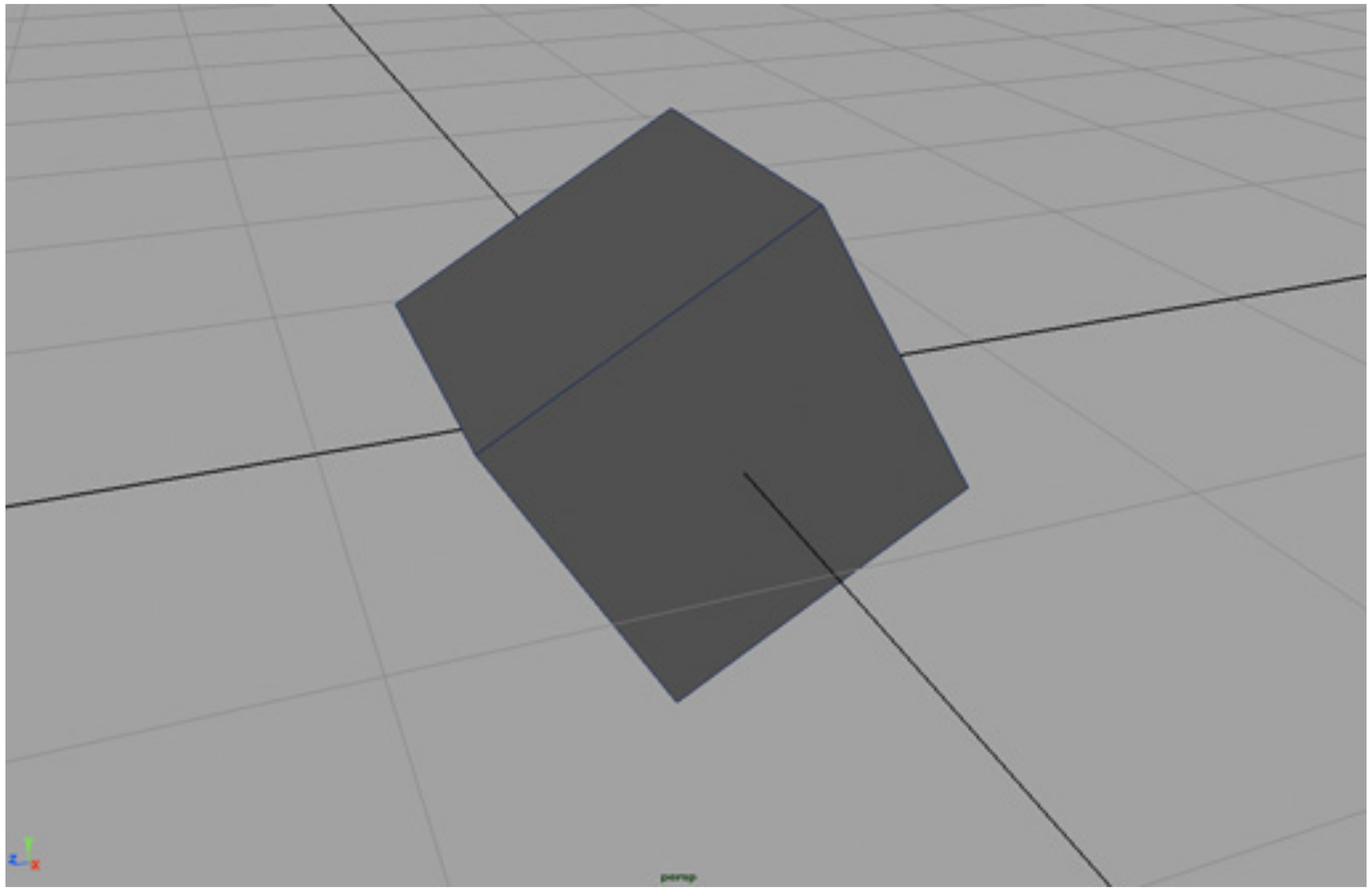
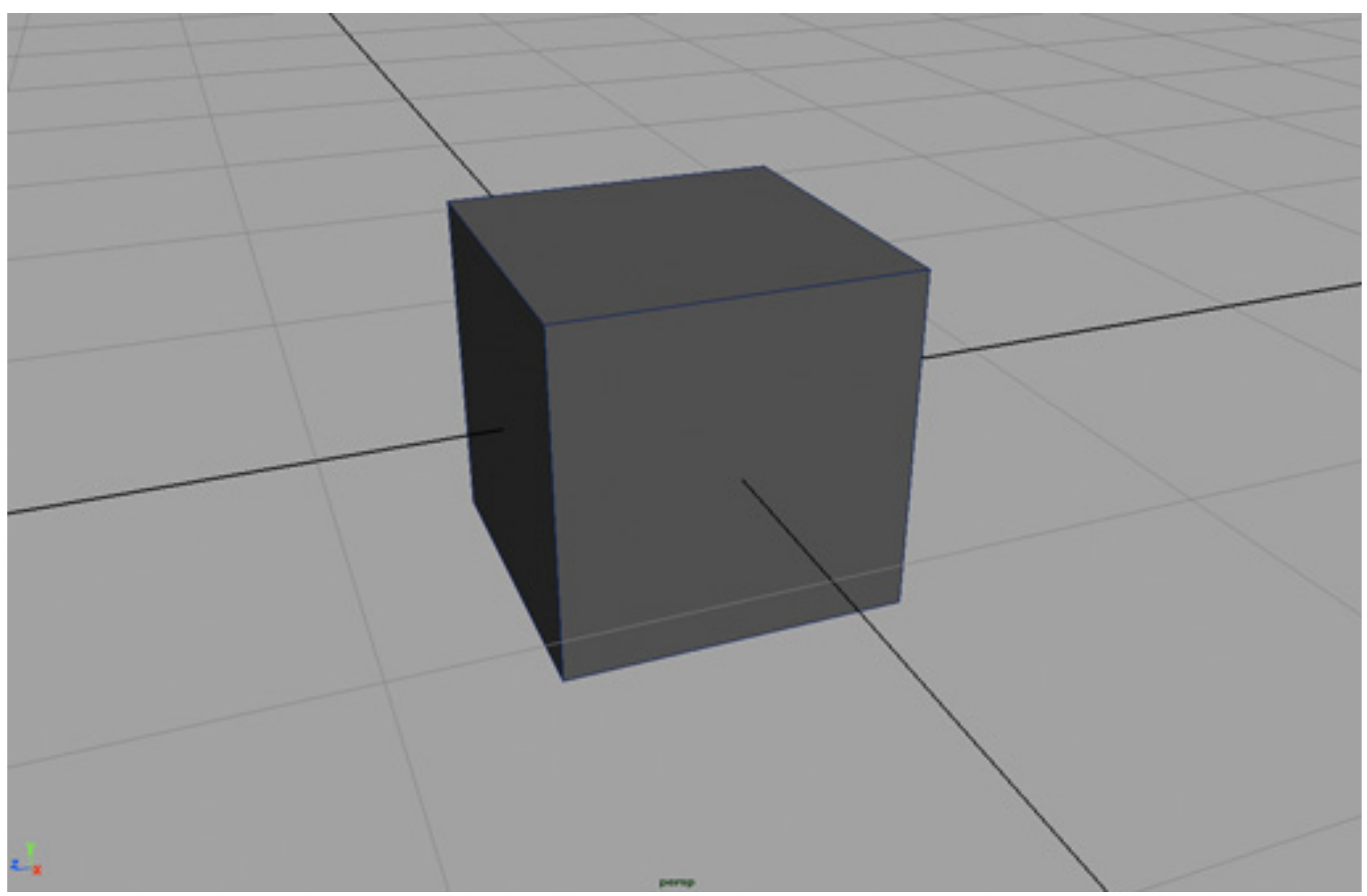


Rotation about y axis

$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} \cos\theta & 0 & \sin\theta & 0 \\ 0 & 1 & 0 & 0 \\ -\sin\theta & 0 & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$



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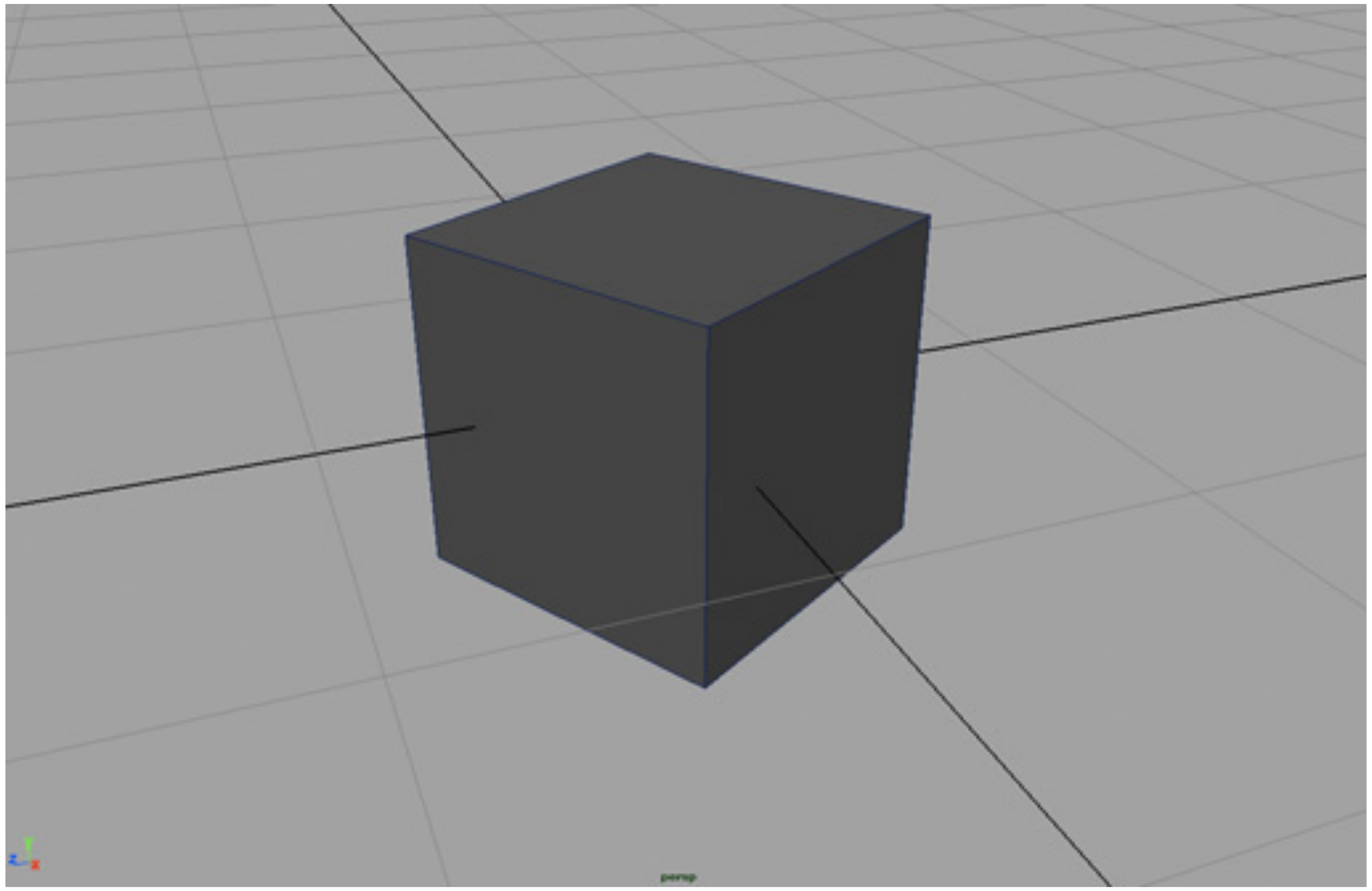


Rotations around an arbitrary axis

- Tricky - many ways to describe them:
  - Euler angles: 3 rotations about 3 axes
  - Axis-angle
  - Quaternions
- Simplest conceptually: indirectly specify via coordinate frame transformations.
  - We did something like this when finding a camera basis!

Rotations around an arbitrary axis

- If you do all that... you get a formula you can find on wikipedia:

  - This is the formula for rotation of angle θ around an unit-length axis $\begin{bmatrix} u_x & u_y & u_z \end{bmatrix}^T$:

$$R = \begin{bmatrix} \cos\theta + u_x^2\left(1-\cos\theta\right) & u_x u_y\left(1-\cos\theta\right) - u_z\sin\theta & u_x u_z\left(1-\cos\theta\right) + u_y\sin\theta \\ u_y u_x\left(1-\cos\theta\right) + u_z\sin\theta & \cos\theta + u_y^2\left(1-\cos\theta\right) & u_y u_z\left(1-\cos\theta\right) - u_x\sin\theta \\ u_z u_x\left(1-\cos\theta\right) - u_y\sin\theta & u_z u_y\left(1-\cos\theta\right) + u_x\sin\theta & \cos\theta + u_z^2\left(1-\cos\theta\right) \end{bmatrix}$$

Transforming Points vs Directions

- Problem 1

Transforming Normals

- Problem 2

Transforming points and vectors

- Recall distinction points vs. vectors
  – vectors are just offsets (differences between points)
  – points have a location
    - represented by vector offset from a fixed origin
- Points and vectors transform differently
  – points respond to translation; vectors do not

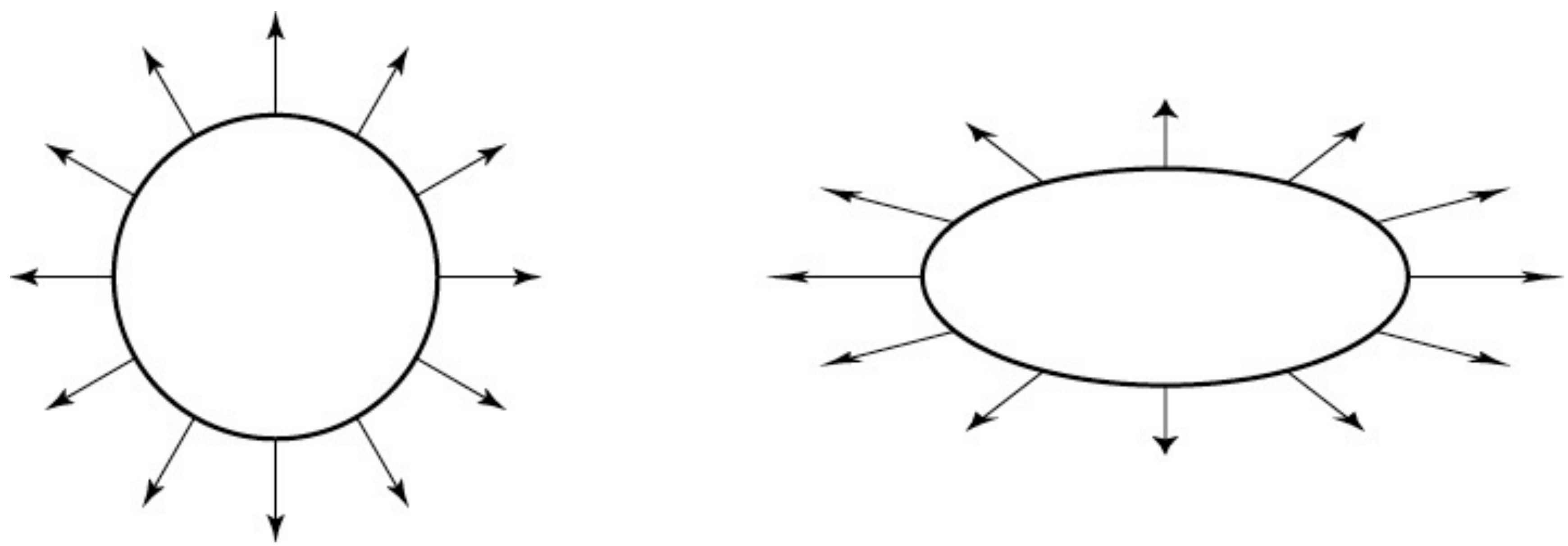
$$\mathbf{v} = \mathbf{p} - \mathbf{q}$$

$$T(\mathbf{x}) = M\mathbf{x} + \mathbf{t}$$

$$\begin{aligned} T(\mathbf{p} - \mathbf{q}) &= M\mathbf{p} + \mathbf{t} - (M\mathbf{q} + \mathbf{t}) \\ &= M(\mathbf{p} - \mathbf{q}) + (\mathbf{t} - \mathbf{t}) = M\mathbf{v} \end{aligned}$$

Transforming normal vectors

- Transforming surface normals
  - –differences of points (and therefore tangents) transform OK
  - –normals do not --> use inverse transpose matrix



have: $\mathbf{t} \cdot \mathbf{n} = \mathbf{t}^T \mathbf{n} = 0$
want: $M\mathbf{t} \cdot X\mathbf{n} = \mathbf{t}^T M^T X \mathbf{n} = 0$
so set $X = (M^T)^{-1}$
then: $M\mathbf{t} \cdot X\mathbf{n} = \mathbf{t}^T M^T (M^T)^{-1} \mathbf{n} = \mathbf{t}^T \mathbf{n} = 0$

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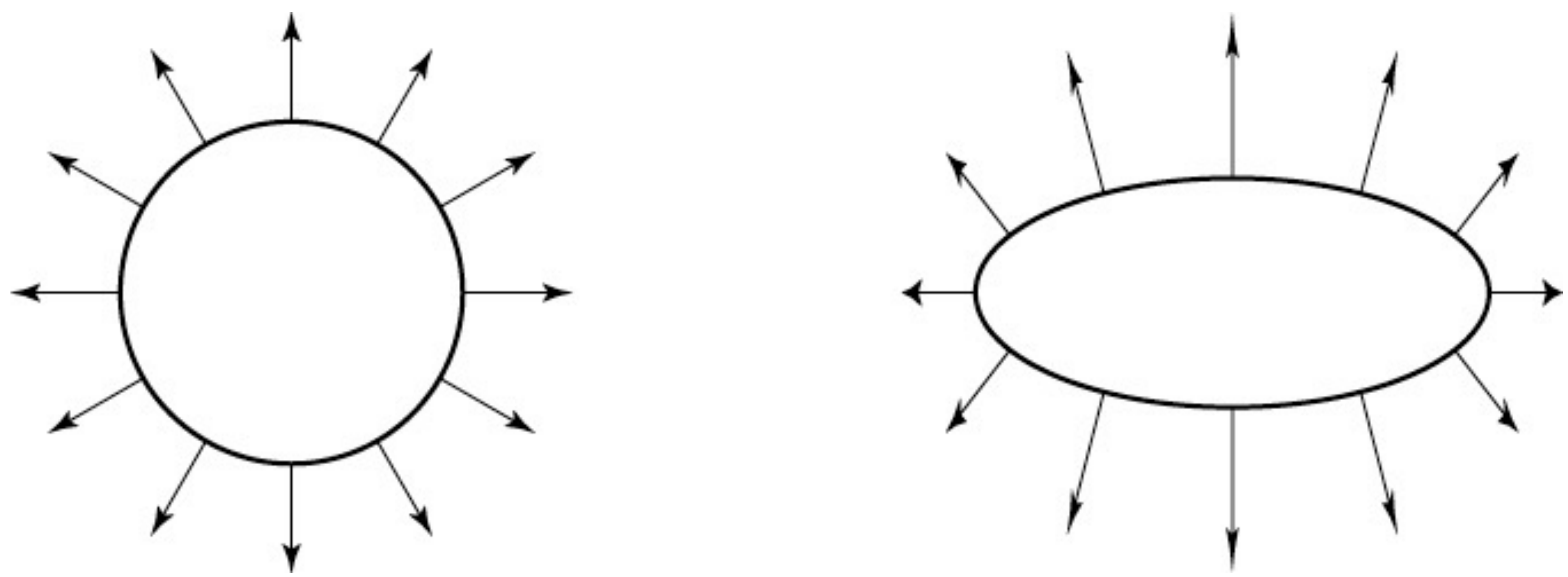

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