Translation: Impossible!

Solution 1: A transformation is now a 2-tuple with a matrix and a vector: $T = (M, \vec{t})$

Applying the transformation is done by:

$$T(\vec{x}) = \boxed{M\vec{x} + \vec{t}}$$

Composing transformations $T_1 \circ T_2$ is... annoying.

$$M_1 (M_2 \vec{x} + \vec{t}_2) + \vec{t}_1 = \underbrace{M_1 M_2}_{M'} + \underbrace{(M_1 \vec{t}_2 + \vec{t}_1)}_{\vec{t}\,'}$$

Solution 2: An Elegant Hack.

$$\begin{bmatrix} x + t_x \\ y + t_y \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

$$M \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} + \begin{bmatrix} t_x \\ t_y \\ 1 \end{bmatrix} = \begin{bmatrix} M & & t_x \\ & & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

Homogeneous Coordinates: represent the 2D point $\begin{bmatrix} x \\ y \end{bmatrix}$ as $\begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$.

Affine Transformation matrix:

2x2 linear transformation

translation

$$\begin{bmatrix} a_{11} & a_{12} & t_x \\ a_{21} & a_{22} & t_y \\ 0 & 0 & 1 \end{bmatrix}$$

$[0 \;\; 0 \;\; 1]$

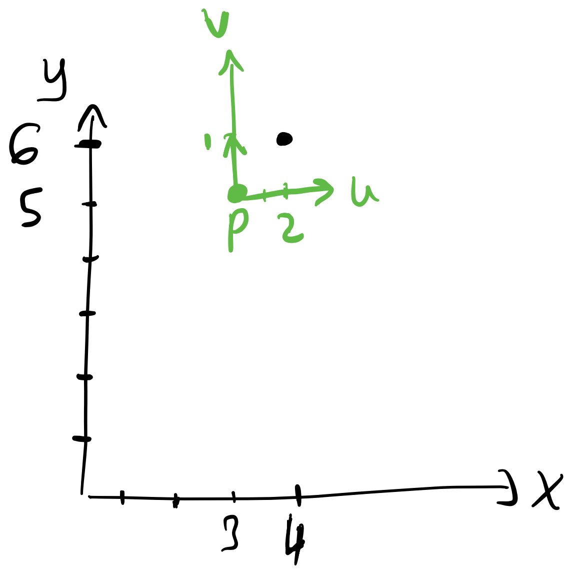
Fact: composition $(T_1 \circ T_2 = T_1 T_2)$ just works! (HW2: Prove it!)

Change-of-~~Basis~~ Frame View:

An affine transformation defines a coordinate frame (origin + basis).

$$\vec{x} = \vec{p} + u\,\vec{u} + v\,\vec{v}$$

$$\begin{bmatrix} 4 \\ 6 \\ 1 \end{bmatrix} = \begin{bmatrix} 1/2 & 0 & 3 \\ 0 & 1 & 5 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$$

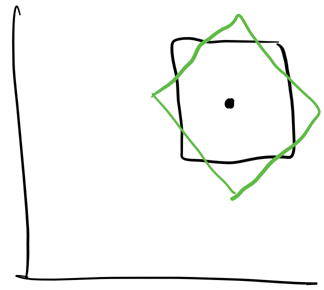


If the basis is orthonormal, this makes a rigid transformation.

$$\begin{bmatrix} Q & \vec{t} \\ 0\ 0 & 1 \end{bmatrix} :$$

- does not change shape or scale.
- May rotate, translate, and reflect.

Similarity Transformations



1. Make the thing easy — change to a frame where the thing is easy
2. Do the easy thing — do easy thing
3. Undo step 1. — return to original frame.

$$F = \begin{bmatrix} \vec{u} & \vec{v} & \vec{p} \end{bmatrix}$$, where desired transform is T

$$F\,T\,F^{-1}(\vec{x})$$

Inverses

Algebraic inverse works, but sometimes it's simpler.

- $\text{translate}(\vec{t})^{-1} = \text{translation}(-\vec{t})$
- $\text{Rotate}(\theta)^{-1} = \text{rotate}(-\theta)$
- $\text{scale}(s_x, s_y)^{-1} = \text{scale}\left(\frac{1}{s_x}, \frac{1}{s_y}\right)$
- $\text{Rigid}(Q, \vec{t})^{-1} =$

Q is orthonormal

$$Q^{-1} = Q^T$$

$$\begin{bmatrix} Q & \vec{t} \\ 0\;0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} Q^T & -Q^T\vec{t} \\ 0 & 1 \end{bmatrix}$$