Computer Graphics

Lecture 15
2D Linear Transformations

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2D Linear Transformations

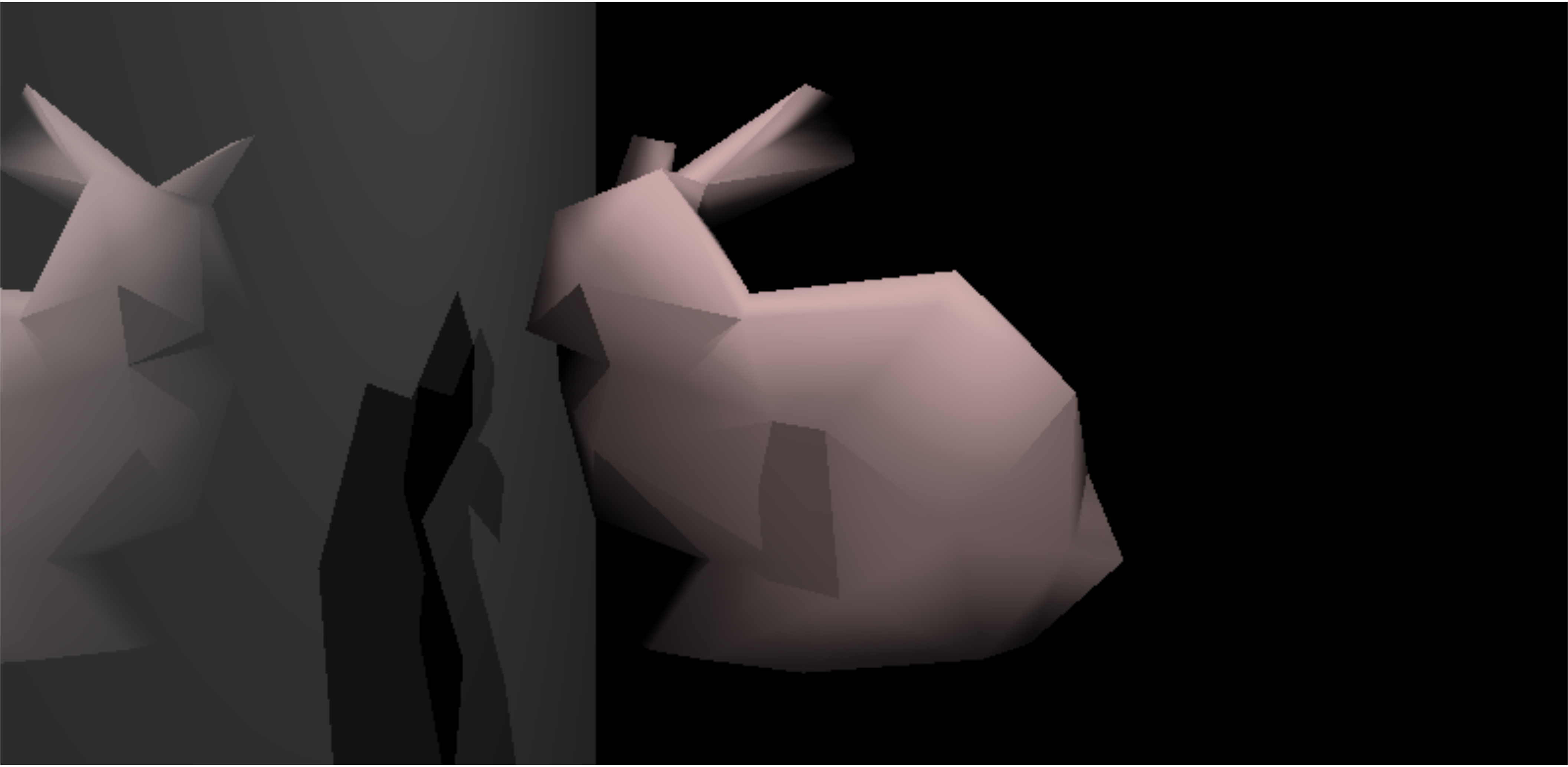
Announcements

- HW2 due tomorrow!?
- A2 due Monday

Goals

- Have intuition for matrices as
  - Linear functions that map points from one place to another in space.
  - Basis-change machines that convert coordinates expressed in some basis into the canonical basis.
- Know how to construct 2D matrices that perform uniform and nonuniform scaling, reflection, and rotation.
- Know some properties of linear transformations:
  - Linearity, closure under composition, associativity, non-commutativity

Situation: Bunny is sad.



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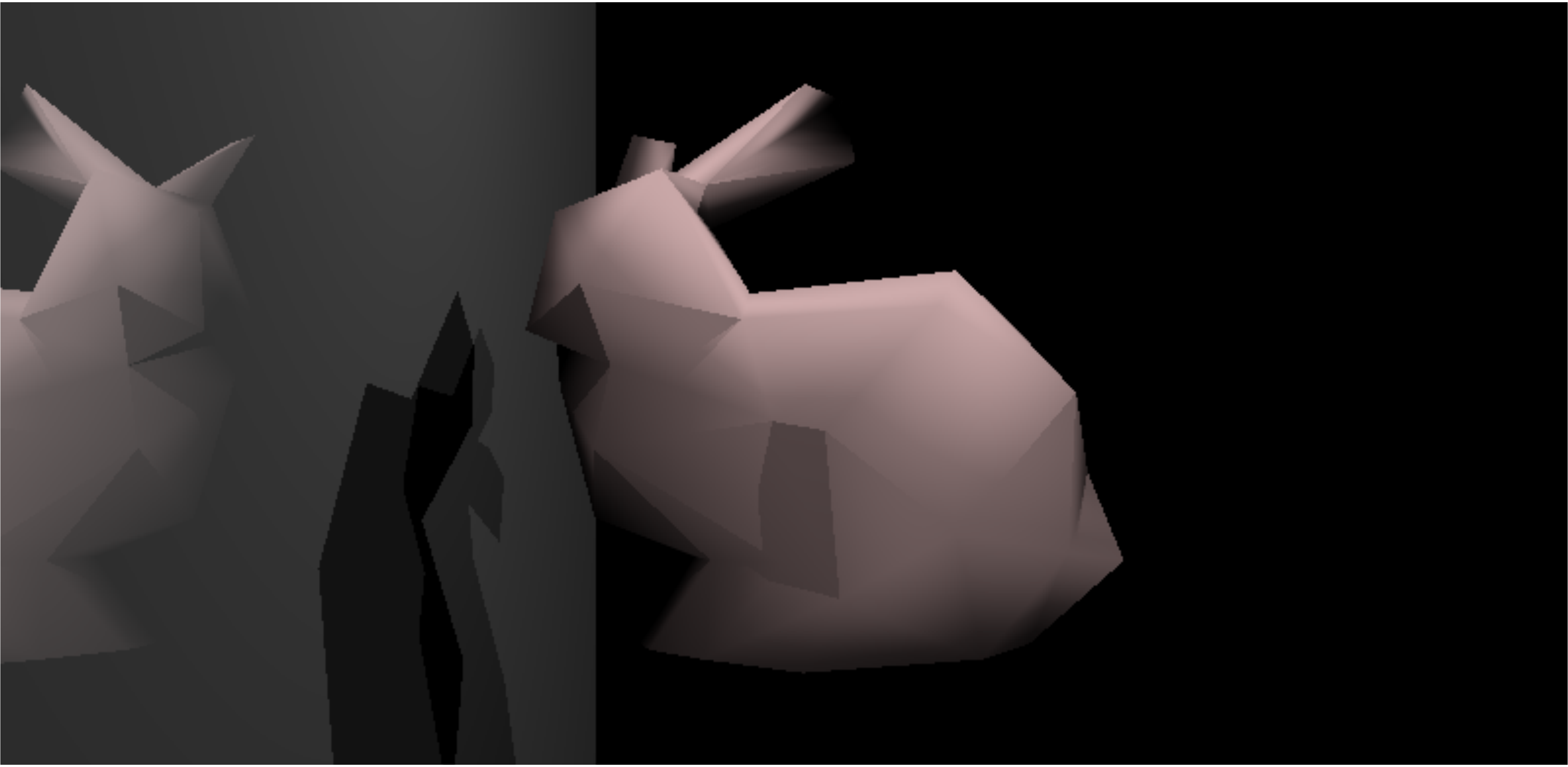

Bunny is sad because it can't move.

Today: Make bunny happy

- How can we manipulate objects in the scene to
  - put them in the right position?
  - scale them to the right size?
  - orient them in the right direction?

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Our answer: matrices.

Geometric Transformations

- To the notes! (This is the notes!)

Goal: a function (mapping) that specifies new geometry given old geometry.

$$T : \mathbb{R}^2 \to \mathbb{R}^2 \text{ in 2D, or } \mathbb{R}^3 \text{ for 3D.}$$

Simple example:

$$T(\vec{x}) = \vec{x} + \vec{t}$$

We will restrict ourselves to Linear T's (for now)

Meaning: $x' = ax + by$
$y' = cx + dy$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

↑

Some Review on Matrices

rows ↓ columns ↓

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 3 & 4.5 & 0.5 \end{bmatrix}$$ is a 2×3 matrix

"by"

$m \times n$

a_{ij} is the ith row, jth column

Matrix-Vector Multiplication: Two Views

$$\begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{bmatrix} = \begin{bmatrix} \text{---} & r_1 & \text{---} \\ \text{---} & r_2 & \text{---} \\ & \vdots & \\ \text{---} & r_m & \text{---} \end{bmatrix} \begin{bmatrix} | \\ \vec{x} \\ | \end{bmatrix}$$

$y_i = r_i \cdot \vec{x}$

$$\begin{bmatrix} | \\ \vec{y} \\ | \end{bmatrix} = \begin{bmatrix} | & | & & | \\ c_1 & c_2 & \cdots & c_n \\ | & | & & | \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

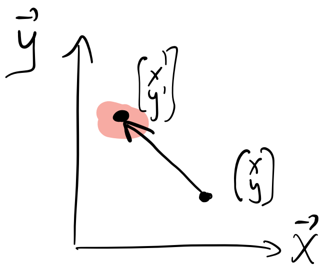
$\vec{y} = x_1 \vec{c}_1 + x_2 \vec{c}_2 + \cdots x_n \vec{c}_n$

Geometric Interpretation

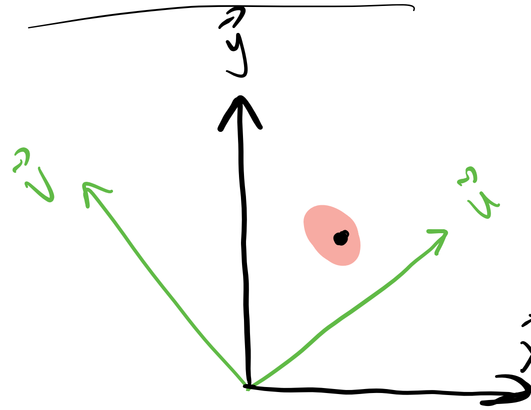
$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} | & | \\ u & v \\ | & | \end{bmatrix} \begin{pmatrix} u \\ v \end{pmatrix}$$

Mapping: $\mathbb{R}^2 \to \mathbb{R}^2$



Change of Basis



Properties of Matrix Multiplication/Linear Transformations

Linear: $Tx + Ty = T(x+y)$ and $aTx = T(ax)$

Associative: $ABC = (AB)C = A(BC)$

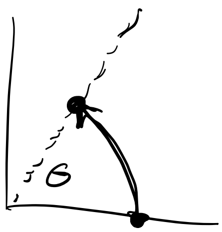
Not Commutative: $AB \neq BA$

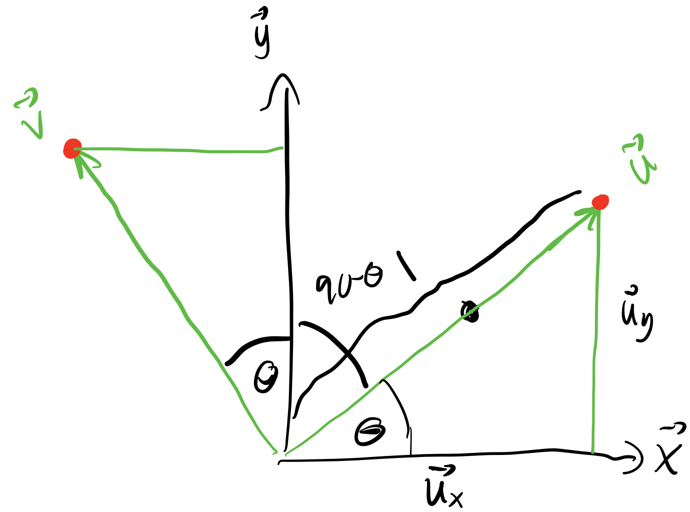
Closed under composition: $(S \circ T) = ST$ is still a linear tx

Right-to-left application: ABx

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} | & | \\ \vec{u} & \vec{v} \\ | & | \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix}$$

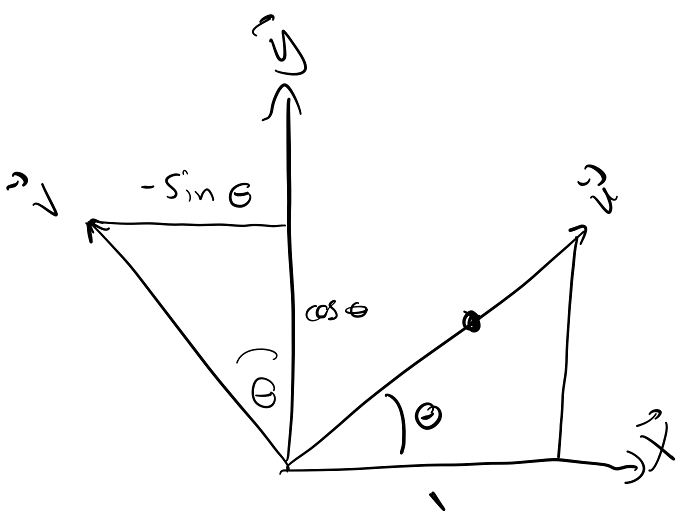
Rotation?





$$\vec{u} = \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix} \qquad \vec{v} = \begin{bmatrix} -\sin\theta \\ \cos\theta \end{bmatrix}$$

$$R_\theta = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$



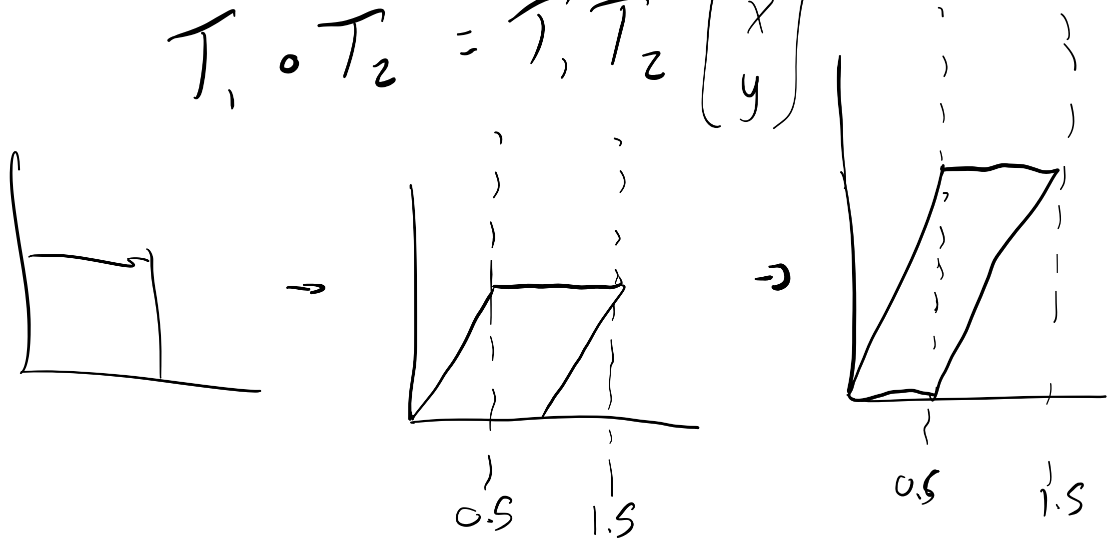
$$\vec{u} = \begin{matrix} \cos\theta \\ \sin\theta \end{matrix}$$

$$\vec{v} = \begin{matrix} -\sin\theta \\ \cos\theta \end{matrix}$$

Composition

$$T_1 = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \qquad T_2 = \begin{bmatrix} 1 & 0.5 \\ 0 & 1 \end{bmatrix}$$

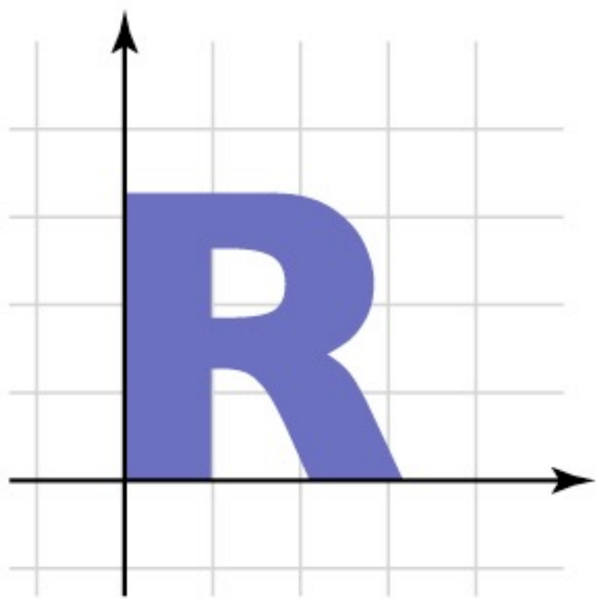
$$T_1 \circ T_2 = T_1 T_2 \begin{bmatrix} x \\ y \end{bmatrix}$$

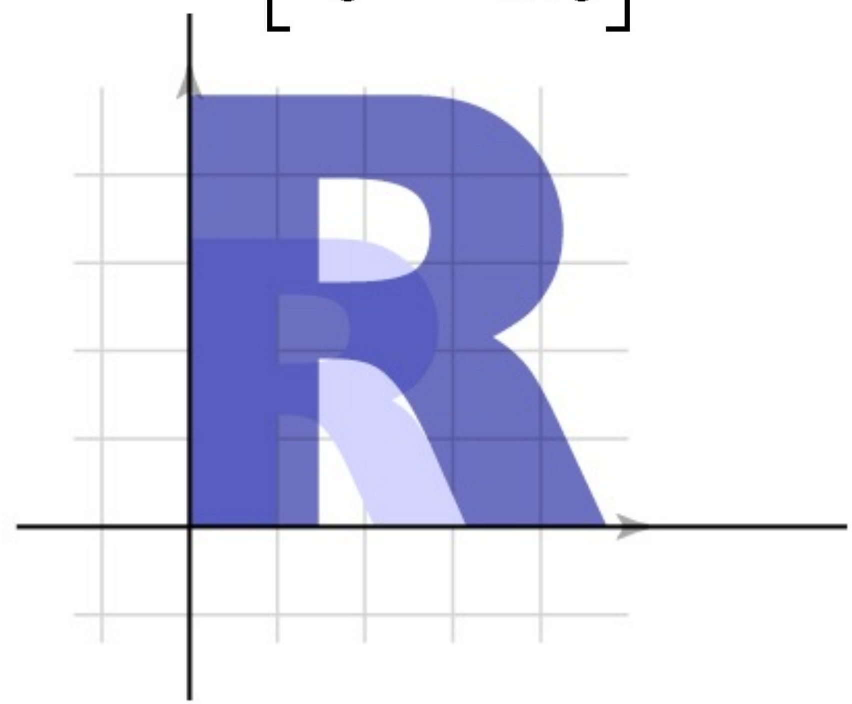


Linear transformation gallery

- Uniform scale $$\begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} sx \\ sy \end{bmatrix}$$

$$\begin{bmatrix} 1.5 & 0 \\ 0 & 1.5 \end{bmatrix}$$

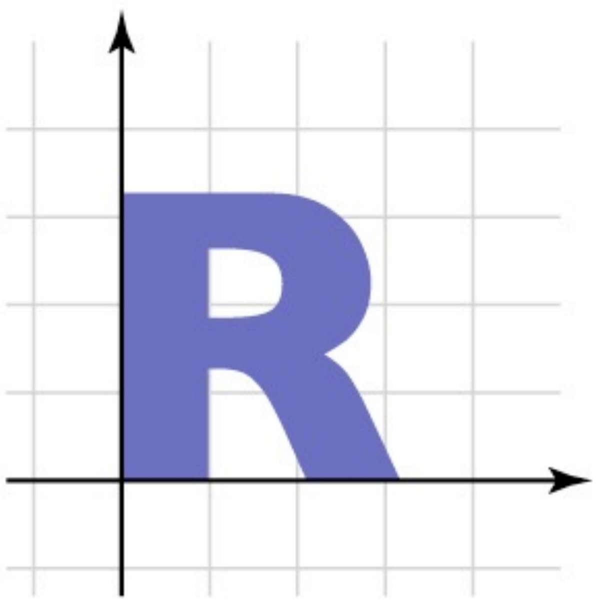


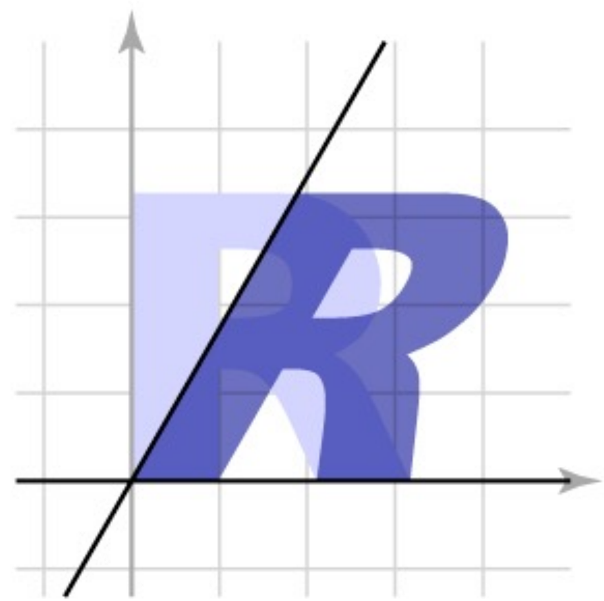


Linear transformation gallery

- Shear $\begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x + ay \\ y \end{bmatrix}$

$$\begin{bmatrix} 1 & 0.5 \\ 0 & 1 \end{bmatrix}$$



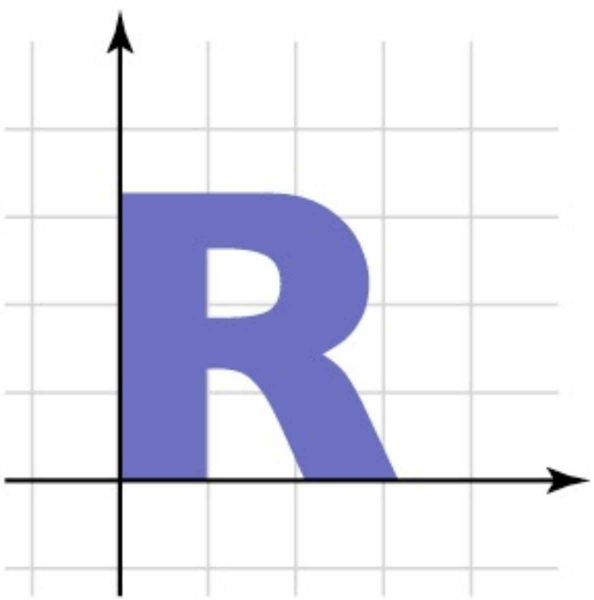


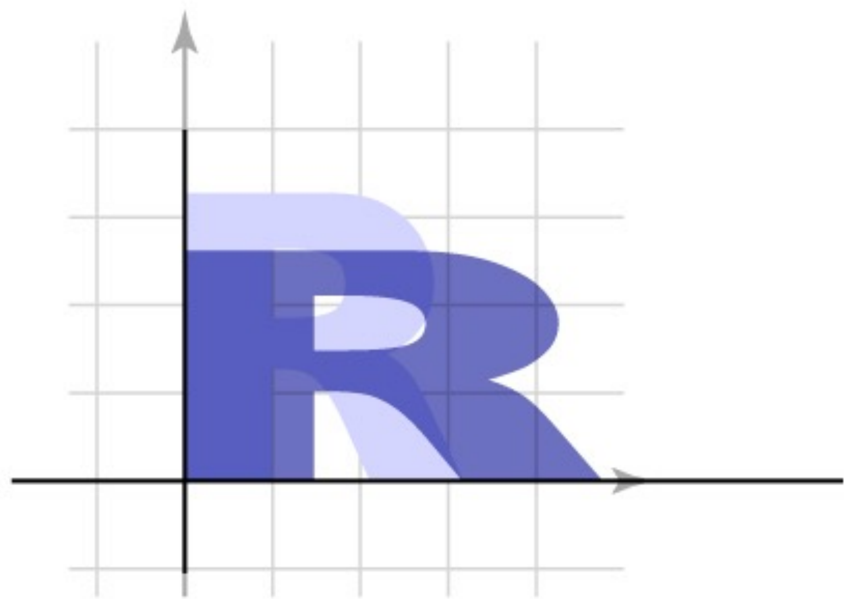
Linear transformation gallery

- Nonuniform scale

$$\begin{bmatrix} s_x & 0 \\ 0 & s_y \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} s_x x \\ s_y y \end{bmatrix}$$

$$\begin{bmatrix} 1.5 & 0 \\ 0 & 0.8 \end{bmatrix}$$

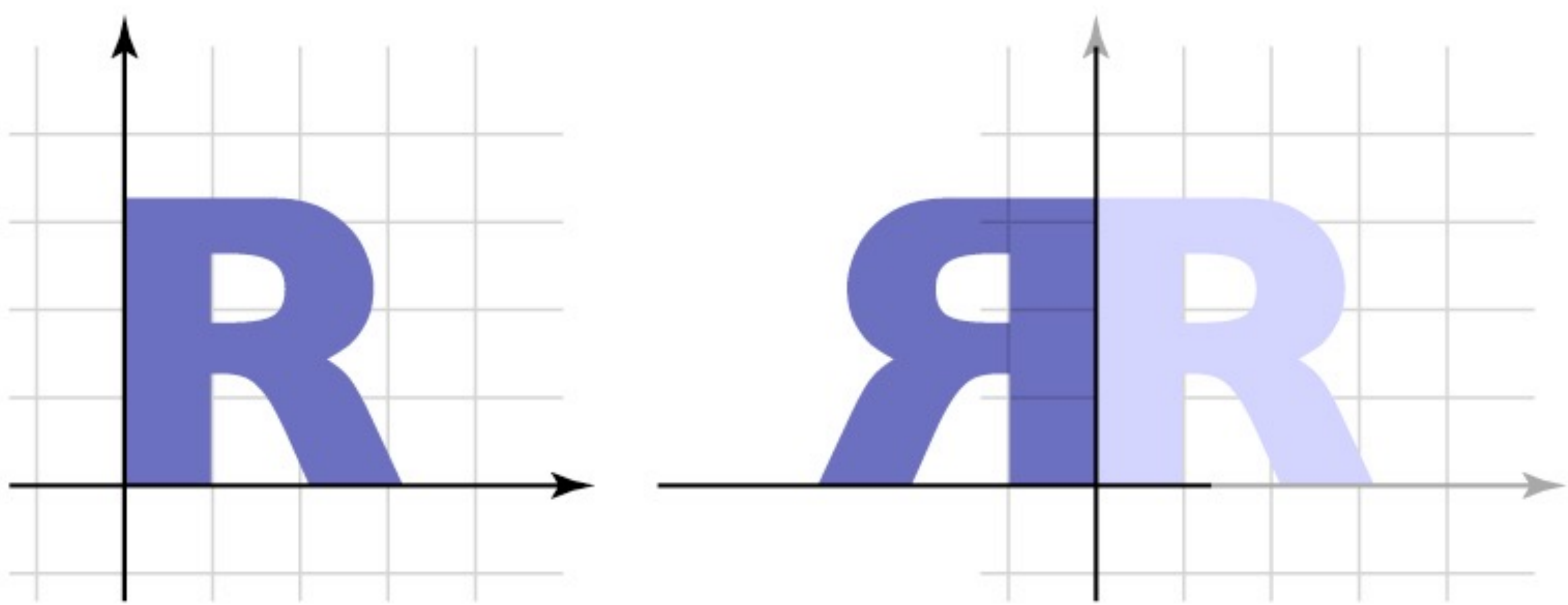




Linear transformation gallery

- Reflection
  - can consider it a special case of nonuniform scale

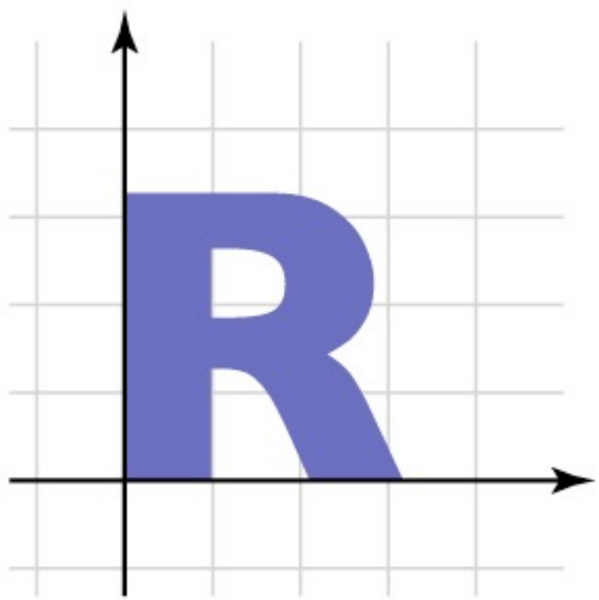
$$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

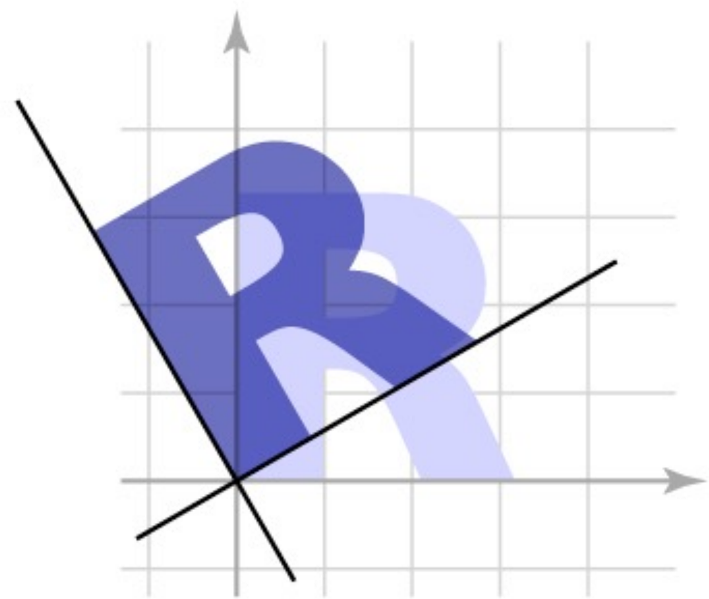


Linear transformation gallery

- Rotation $$\begin{bmatrix}\cos\theta & -\sin\theta \\ \sin\theta & \cos\theta\end{bmatrix}\begin{bmatrix}x \\ y\end{bmatrix} = \begin{bmatrix}x\cos\theta - y\sin\theta \\ x\sin\theta + y\cos\theta\end{bmatrix}$$

$$\begin{bmatrix}0.866 & -0.5 \\ 0.5 & 0.866\end{bmatrix}$$





2D Matrix Transformations: Properties

- linear
- closed under composition
- associative
- not commutative
- applied right-to-left