Mechanical (Shipbuilder's) approach

-> smoothness from splines

-> control from pegs

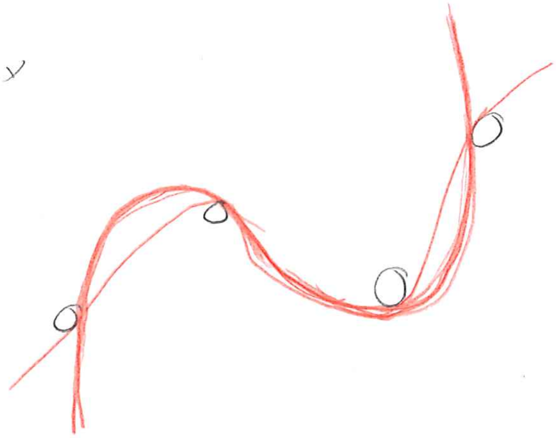
Piecewise linear

first-last joining

Math approach:

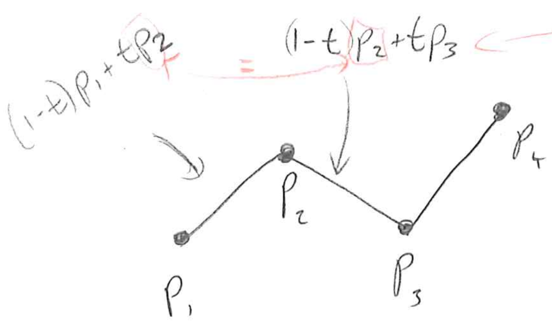
-> smoothness from low-deg poly

-> control from control points



Start simple: Piecewise linear

(i.e. degree-1 polynomials!)



Can adjust to use a "universal" t, but easiest to consider each segment alone

We'll fix two things about this:

- it's not smooth
- the equations don't generalize elegantly or give other types of control

Linear interpolation, or: a parametric line segment:

$$f(u) = (1-u)\vec{p}_0 + u\vec{p}_1$$

Specified via endpoints p_0, p_1

Convention: u is like t, but ranges from 0 to 1 only

(we'll omit $\rightarrow$ vector symbols, assume all points are ND vectors)

$$f(u) = a_0 + u(a_1)$$

Specified by start (a_0) and vector-pointing from start to end (a_1)

View this as a degree-1 polynomial:

$f(u)$ $u^0 a_0 + u^1 a_1$; can also write this as a dot product:

$$f(u) = \vec{u} \cdot \vec{a} \quad \begin{bmatrix} 1 & u \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \end{bmatrix}$$

Convert between these two - straightforward:

$$p_0 = \text{beginning} = f(0) = \begin{bmatrix} 1 & u \end{bmatrix} \cdot \begin{bmatrix} a_0 \\ a_1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \end{bmatrix} = a_0$$

Current value of u

$$\boxed{p_0 = a_0}$$

$$p_1 = \text{end} = f(1) = \begin{bmatrix} 1 & u \end{bmatrix} \cdot \begin{bmatrix} a_0 \\ a_1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \end{bmatrix} = a_0 + a_1$$

$$p_1 = a_0 + a_1, \text{ so } a_1 = p_1 - a_0 = p_1 - p_0$$

$$\boxed{a_1 = p_1 - p_0}$$

"control points"

Canonical form/coefficients

Even more structure:

$$\begin{bmatrix} p_0 \\ p_1 \end{bmatrix} = \begin{array}{cc} 1 & u^1 \\ \end{array}$$
$$\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \end{bmatrix}$$

$$\vec{p} = C \quad \vec{a}$$

Solve by inverting C: if $B = C^{-1}$

$$\vec{p} = C\vec{a}$$
$$B\vec{p} = BC\vec{a}$$
$$\vec{a} = B\vec{p}$$

Why did we want $\vec{a}$?

Because $f(u) = \vec{u} \cdot \vec{a}$

Canonical form, easy to compute

Ok but we already knew how to draw lines.

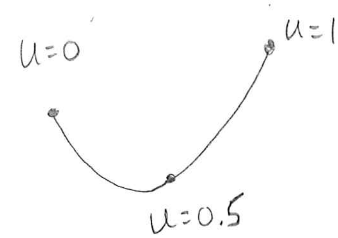
Here's a quadratic, using the same machinery:

$$f(u) = a_0 u^0 + a_1 u^1 + a_2 u^2 = \vec{a} \cdot \vec{u}$$

These aren't geometrically intuitive ways to specify a parabola (segment)

We want to describe it via 3 points: start, middle, end:

Same procedure:



$$\begin{aligned} P_0 &= f(0) = 1a_0 + \boxed{0^1} a_1 + \boxed{0^2} a_2 \\ P_1 &= f(1) = 1a_0 + \boxed{0.5^1} a_1 + \boxed{0.5^2} a_2 \\ P_2 &= f(2) = 1a_0 + \boxed{1^1} a_1 + \boxed{1^2} a_2 \end{aligned}$$

Matrix form:

$$\begin{bmatrix} P_0 \\ P_1 \\ P_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0.5 & 0.25 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} \text{ or } \vec{p} = C\vec{a}$$

Given $\vec{p}$, find $\vec{a}$ so we can evaluate $f(u) = \vec{u} \cdot \vec{a}$ for any u

$\vec{p} = C\vec{a}$, invert $C^{-1} = B$

$$\vec{a} = B\vec{p} \qquad C^{-1} = B = \begin{bmatrix} 1 & 0 & 0 \\ -3 & 4 & -1 \\ 2 & -4 & 2 \end{bmatrix}$$

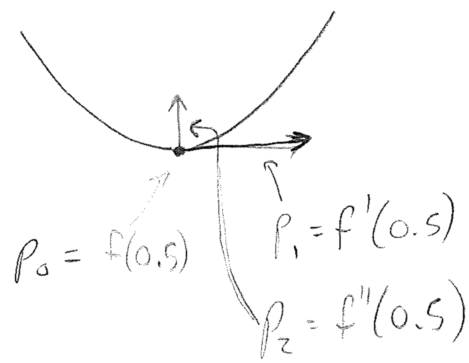
Another type of control point: specify <u>derivative</u> at a pt.

Example: specify midpoint, 1st, and 2nd derivatives at midpt.

$$P_0 = f(0.5) = a_0 + 0.5^1 a_1 + 0.5^2 a_2$$

$$P_1 = f'(0.5) = a_1 + 0.5 \cdot 2 a_2$$

$$P_2 = f''(0.5) = 2 a_2$$



$$C = \begin{pmatrix} 1 & 0.5 & 0.25 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \end{pmatrix}$$

derivatives?

$$f(u) = a_0 + a_1 u + a_2 u^2$$

$$f'(u) = a_1 + 2 a_2 u$$

$$f''(u) = 2 a_2$$

Again, $\vec{p} = C\vec{a}$, so $\vec{a} = C^{-1}\vec{p}$