Announcements - 2/24

- Create FP repo by tonight (midnight)
- A3 due tonight (10pm)
- FP milestone 1 due Monday
- Complete line lab by Friday night (10pm)

Announcements - 2/26

- Submit line lab by tonight
- FP milestone Monday

Lines: ✓

Modeling:

$y = mx+b$
$ax+by+c=0$ } → (implicit)

$\vec{p} + t\vec{d}$ } → (parametric)

Rendering: Bresenham/midpoint alg.

Curves:

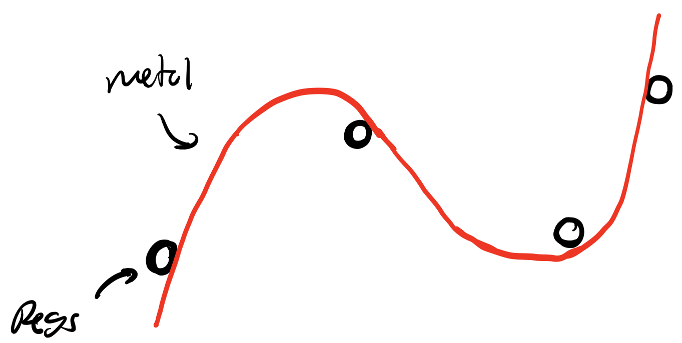
Modeling: piecewise (low-degree) polynomial pieces

Rendering: GOTO modeling
piecewise linear segments

Modeling Curves: The old-fashioned way, used in ship building

→ Control: pegs

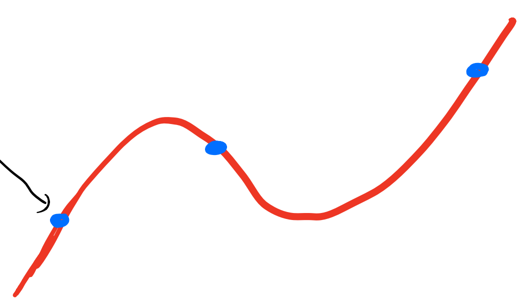
→ Smoothness: physics



Modeling Curves: The Math Way

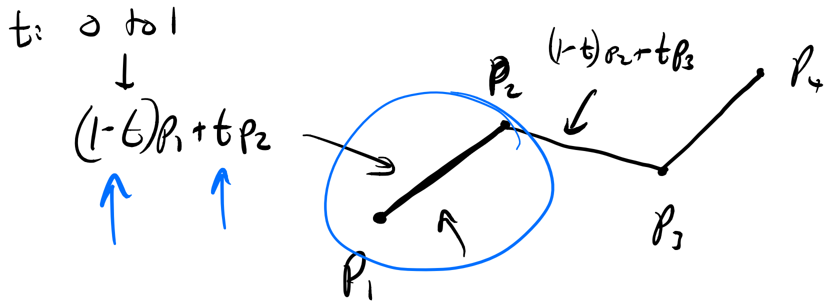
→ control: control points

→ smoothness:
low-deg polynomial



Simplest Case: degree-1 polynomials

a.k.a. line segments



Two problems:

(1) not smooth - need higher-degree polynomials

(2) approach doesn't generalize nicely

Reminder: linear interpolation ("lerp")

Have: $f(u) = (1-u)p_0 + u p_1$

Problem: not obviously polynomial
Solution: convert to something that is.

Notation:

- u is like t, but by convention varies only from 0 to 1.
- p_0, p_1 are points (of any dimension!)

Want: $f(u) = u^0 a_0 + u^1 a_1 = \begin{bmatrix} 1 & u \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \end{bmatrix} = \vec{u} \cdot \vec{a}$

Task: find p in terms of a

$$\begin{array}{llcc}
 & & 1\ (u^0) & u^1 \\
\text{start} & p_0 = f(0) = & \boxed{1}\, a_0 + & \boxed{0}\, a_1 \\
\text{end} & p_1 = f(1) = & \boxed{1}\, a_0 + & \boxed{1}\, a_1
\end{array}$$

Matrix Form:

$$\begin{bmatrix} p_0 \\ p_1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \end{bmatrix}$$

Control points $\rightarrow$ $\vec{p} = \underset{\text{constraint matrix}}{C} \ \vec{a}$ $\leftarrow$ coefficients

To draw curves, we have p, want a.

$$\boxed{C^{-1}\vec{p} = \vec{a}}$$

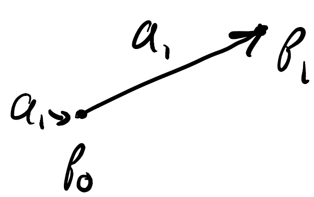
Recall: given $\vec{a}$ and a parameter value u, $f(u) = \underset{[1 \ u]}{\vec{u}} \cdot \underset{\begin{bmatrix} a_0 \\ a_1 \end{bmatrix}}{\vec{a}}$

Let $\boxed{B = C^{-1}}$. Then:

$$f(u) = \underset{[1 \ u]}{\vec{u}^T} \vec{a} = \vec{u}^T \underset{\text{Basis matrix}}{B} \underset{\text{ctrl pts}}{\vec{p}}$$

$$\underset{C}{\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}^{-1}} = \underset{B}{\begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}}$$

$$\begin{matrix} p_0 \\ p_1 - p_0 \end{matrix} = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} \begin{matrix} p_0 \\ p_1 \end{matrix}$$

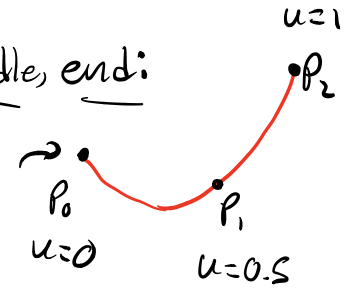


Ok but we already knew how to draw lines.

Let's apply the same machinery to quadratics!

$$f(u) = a_0 \overset{\downarrow}{u^0} + a_1 \overset{\downarrow}{u^1} + a_2 \overset{\overset{\text{new!}}{\downarrow}}{u^2} = \vec{u} \cdot \vec{a}$$

Instead, define control points: beginning, middle, end:



$$\begin{array}{llllll} & & & u^0=1 & u^1 & u^2 \\ \text{Start} & P_0 = f(0) & = & \boxed{1}\, a_0 + & \boxed{0}\, a_1 + & \boxed{0}\, a_2 \\ \text{mid} & P_1 = f(0.5) & = & \boxed{1}\, a_0 + & \boxed{0.5}\, a_1 + & \boxed{0.25}\, a_2 \\ \text{end} & P_2 = f(1) & = & \boxed{1}\, a_0 + & \boxed{1}\, a_1 + & \boxed{1}\, a_2 \end{array}$$

$\vec{p} = C\vec{a}$

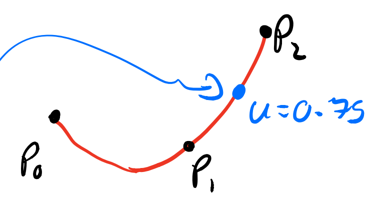
C^{-1}

$\vec{a} = B\vec{p}$

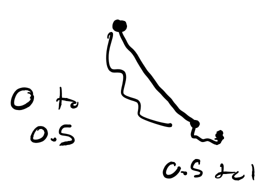
$$C = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0.5 & 0.25 \\ 1 & 1 & 1 \end{pmatrix} \qquad B = C^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -3 & 4 & -1 \\ 2 & -4 & 2 \end{bmatrix}$$

As before: $f(u) = \vec{u}^T B \vec{p}$

$f(0.75)\ [1 \quad 0.75 \quad 0.75^2] \cdot B\vec{p}$



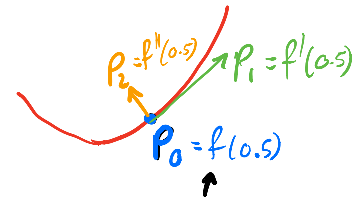
for u=0:.01:1
  draw_pt(f(u))



New kind of control: specify the derivatives at a point.

Example: Quadratic defined by:

- its middle
- its derivative at the middle
- its 2nd derivative at the middle



Pre-compute derivatives:

$$f(u) = a_0 + u a_1 + u^2 a_2$$
$$\rightarrow f'(u) = a_1 + 2u a_2$$
$$\rightarrow f''(u) = 2a_2$$

Constraints:

$$\begin{array}{llllll} & & u^0=1 & & u^1 & & u^2 \\ \text{mid} & P_0 = f(0.5) & = 1\,a_0 & + & 0.5\,a_1 & + & 0.25\,a_2 \\ \text{deriv @mid} & P_1 = f'(0.5) & = 0\,a_0 & + & 1\,a_1 & + & 1\,a_2 \\ \text{2nd deriv @mid} & P_2 = f''(0.5) & = 0\,a_0 & + & 0\,a_1 & + & 2\,a_2 \end{array}$$

$$C = \begin{bmatrix} 1 & 0.5 & 0.25 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \end{bmatrix} \qquad C^{-1} = B = \begin{bmatrix} 1 & 0.5 & 0.125 \\ 0 & 1 & -0.5 \\ 0 & 0 & 0.5 \end{bmatrix}$$

Hold on: P_0 is a vector - how can you make $\vec{p}$?

(1) transpose things carefully, do mat-mat multiplication

or (2) do everything one dim at a time:

$$f(u) = \vec{u}^T B \vec{p}$$

$$\begin{pmatrix} a_{0x} \\ a_{1x} \\ a_{2x} \end{pmatrix} = \begin{bmatrix} & & \\ & B & \\ & & \end{bmatrix}_{3\times 3} \begin{pmatrix} \vec{p}_{0x} \\ \vec{p}_{1x} \\ \vec{p}_{2x} \end{pmatrix}$$