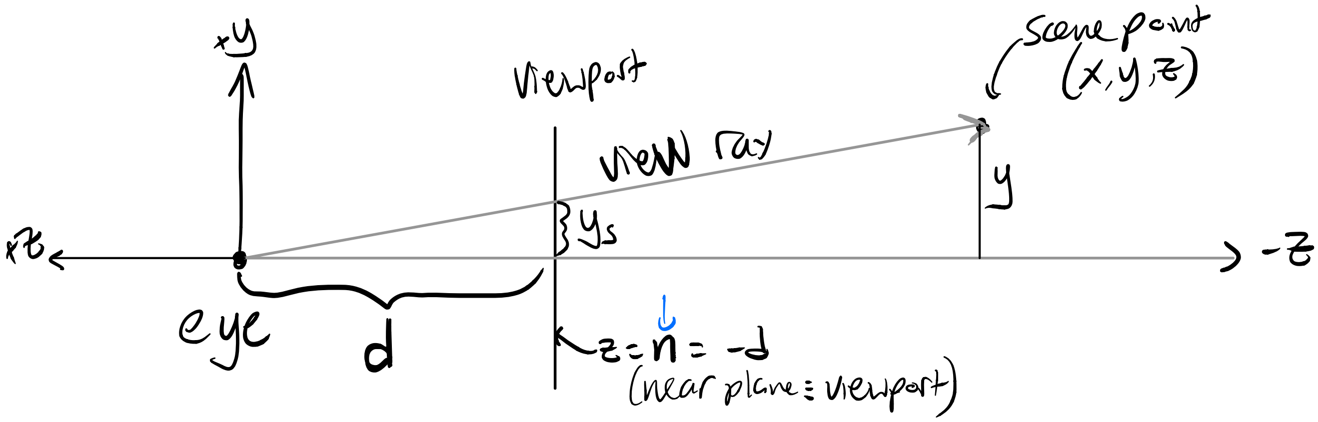
Perspective Viewing



Similar triangles: 

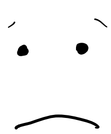
$$y_s = \frac{ny}{z} \qquad\qquad \frac{n}{y_s} = \frac{z}{y}$$

$$x_s = \frac{nx}{z}$$ (by analogous reasoning)

Want:

$$\begin{bmatrix} \frac{nx}{z} \\ \frac{ny}{z} \\ z \\ 1 \end{bmatrix} = \begin{bmatrix} \quad & \quad & \quad & \quad \\ & & & \\ & & & \\ & & & \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

... But you can't always get what you want.



Division by z is not possible using only matmul.

Projective Transformations

Extend our interpretation of homogeneous coords

Before: $\begin{pmatrix} x \\ y \\ z \\ 1 \end{pmatrix}$ Now: $\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix}$ with $w \neq 1$.

What does this even <u>mean</u>?

<u>Definition:</u> $\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix}$ is <u>equivalent</u> to $\begin{bmatrix} x/w \\ y/w \\ z/w \\ w/w = 1 \end{bmatrix}$ if $w \neq 0$

$\sim$

normalized

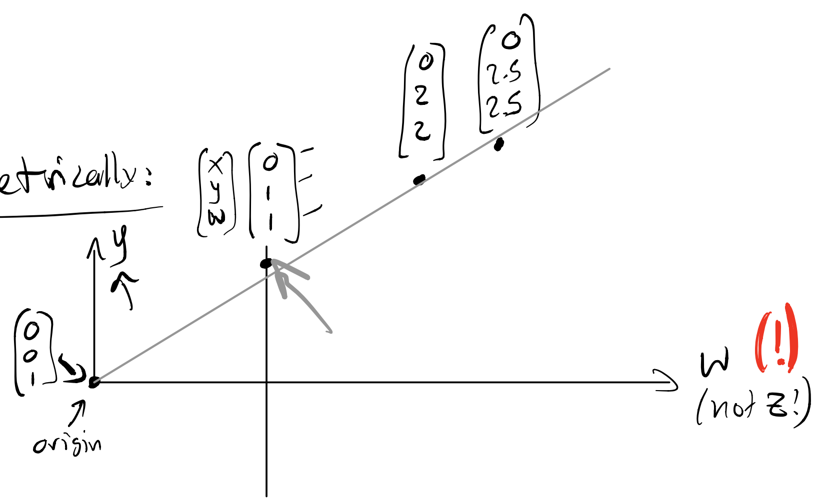
(what if it is? Take CV)

<u>Consequence:</u> $\vec{x} \sim k\vec{x}$ for any $k \neq 0$.

Exercise: Which of the following are the same point as $\begin{bmatrix} 0 \\ 2 \\ 3 \\ 1 \end{bmatrix}$?

~~a~~ $\begin{bmatrix} 0 \\ 8 \\ 12 \\ 3 \end{bmatrix}$ ~~b~~ $\begin{bmatrix} 1 \\ 4 \\ 6 \\ 2 \end{bmatrix}$ ~~c~~ $\begin{bmatrix} 0 \\ 4 \\ 6 \\ 1 \end{bmatrix}$ d ✓ $\begin{bmatrix} 0 \\ -200 \\ -300 \\ -100 \end{bmatrix}$

<u>Geometrically:</u>



Why was w always 1 before?

affine transformation

$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

so far, $w = 1$ always

Break this:
$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix}$$

Projective transformation

$$\begin{bmatrix} \tilde{x} \\ \tilde{y} \\ \tilde{z} \\ \tilde{w} \end{bmatrix} = \begin{bmatrix} a_{11}x + a_{12}y + a_{13}z + a_{14} \\ a_{21}x + a_{22}y + a_{23}z + a_{24} \\ a_{31}x + a_{32}y + a_{33}z + a_{34} \\ \boxed{a_{41}x + a_{42}y + a_{43}z + a_{44}} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

Demo: what can projective xforms do?

$$x' = \frac{\tilde{x}}{\tilde{w}}$$

$$y' = \frac{\tilde{y}}{\tilde{w}}$$

$$z' = \frac{\tilde{z}}{\tilde{w}}$$

Key point: we can shoehorn division in using the normalization of homog. coords.

Recall: we wanted

$$\begin{bmatrix} nx \\ ny \\ \boxed{(n+f)z-fn} \\ z \end{bmatrix} \sim \begin{bmatrix} \frac{nx}{z} \\ \frac{ny}{z} \\ \boxed{n+f-\frac{fn}{z}} \\ 1 \end{bmatrix} \qquad \underbrace{\begin{bmatrix} n & 0 & 0 & 0 \\ 0 & n & 0 & 0 \\ 0 & 0 & n+f & -fn \\ 0 & 0 & 1 & 0 \end{bmatrix}}_{M_{persp}} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

Can't directly carry z through.

Compromise: make it correct at $z=n$ and $z=f$.

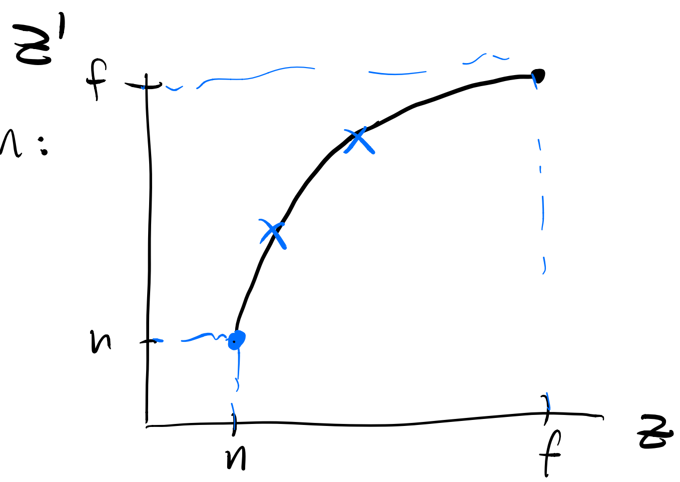
$$z' = n+f-\frac{fn}{z} \quad \text{(presented w/o proof)}$$

Check:

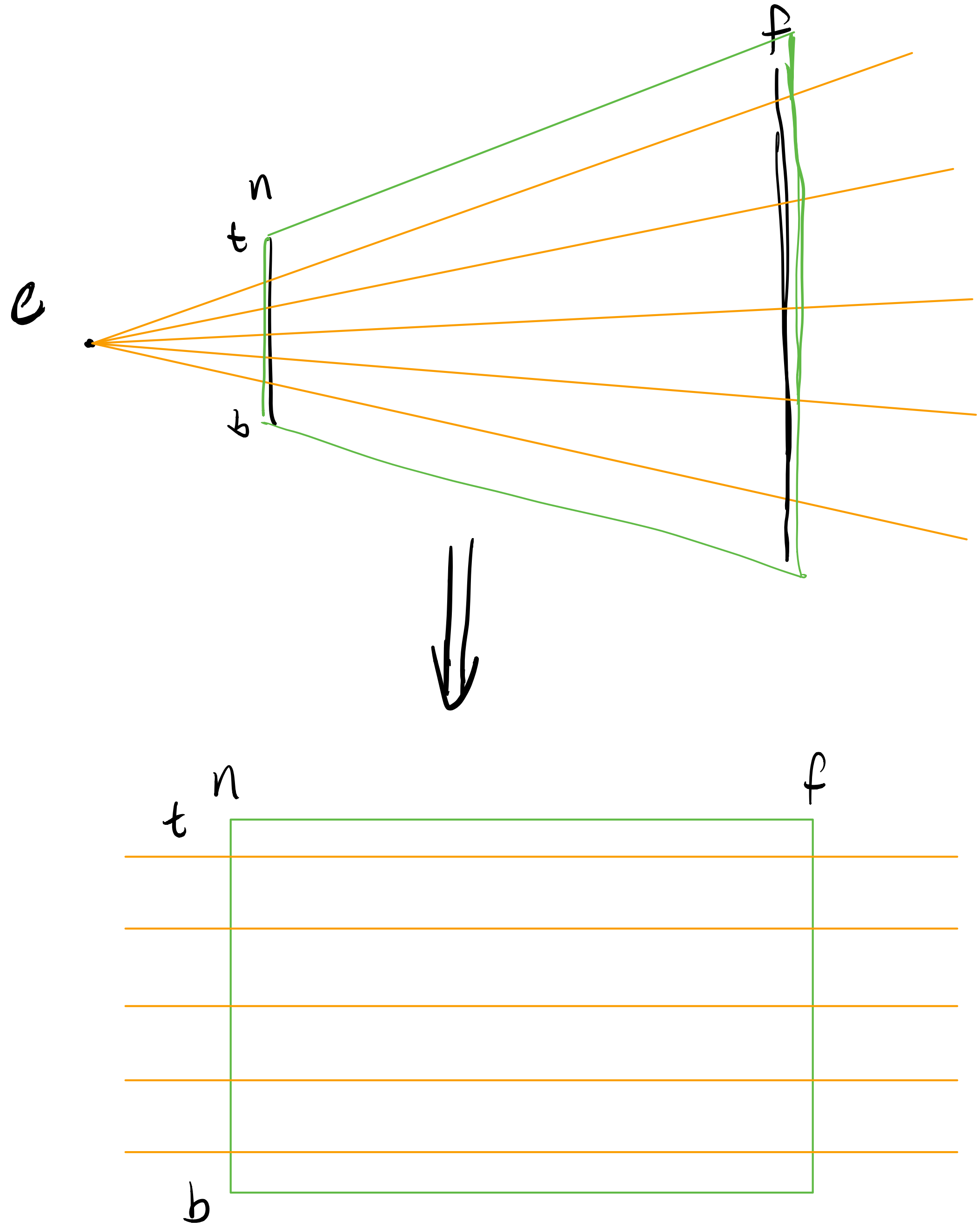
at $z=n$: $\quad n+f-\frac{fn}{n} = n+f-f = \boxed{n}$ ✓
near plane

at $z=f$: $\quad n+f-\frac{fn}{f} = n+f-n = \boxed{f}$ ✓
far plane

In between:



Geometrically (side view)



same as the orthographic view volume!

Reuse M_{ortho} - $\quad M_{proj} = \underbrace{\boxed{M_{ortho}}}_{\text{(last time)}} \underbrace{\boxed{M_{persp}}}_{\text{(above)}}$