What about Perspective Cameras?

Looking side-on at a canonical perspective camera:

- $$(d - d_n - x)$$
- Viewport
- Scene point $(x, y, z)$
- $y_s$
- $y$
- $> -z$ axis

**Exercise:**

What is $y_s$ in terms of $(x, y, z, d)$?

Similar triangles!?

$$\frac{y_s}{d} = \frac{y}{z}, \text{ or } y_s = \frac{dy}{dz}$$

We'd like a matrix that does this:

$$\begin{pmatrix} d x & d y & d z \\ x & y & z \end{pmatrix} = \begin{pmatrix} X \\ Y \end{pmatrix}$$

But no such matrix exists! 😞

Can't divide by $z$. But this isn't easy - more later.
Projective Transformations

As they are, matrices can't tell us where a 3D point will land on the viewport of a perspective camera.

To fix this, we're going to extend our math hack (homogeneous coordinates) to allow w values ≠ 1.

\[
\begin{bmatrix}
X \\
Y \\
Z \\
W
\end{bmatrix}
\sim
\begin{bmatrix}
X/W \\
Y/W \\
Z/W \\
1
\end{bmatrix}
\]

A homogeneous point is considered equivalent to all possible scalings of itself. 

\[ X \sim kX \text{ for any } k \text{ (assume } k \neq 0) \]

Ex 1: Homog. Point Equivalence

Now, a transformation matrix is free to mess with W!

\[
\begin{bmatrix}
\begin{bmatrix}
X' \\
Y' \\
Z'
\end{bmatrix}
= \begin{bmatrix}
a_{11} & a_{12} & a_{13} & a_{14} \\
a_{21} & a_{22} & a_{23} & a_{24} \\
a_{31} & a_{32} & a_{33} & a_{34} \\
a_{41} & a_{42} & a_{43} & a_{44}
\end{bmatrix}
\begin{bmatrix}
X \\
Y \\
Z \\
1
\end{bmatrix}
\end{bmatrix}
\]

Ex 2: Identity Matrix on homogeneous coordinates

These points all need to end up w/ same (x,y) - do this by messing with W!
Recall, we wanted:

\[
\begin{bmatrix}
\frac{dx}{dy} & x
\end{bmatrix}
\begin{bmatrix}
\frac{dy}{dz} & y
\end{bmatrix}
= \begin{bmatrix}
M_{\text{prop}}
\end{bmatrix}
\begin{bmatrix}
X
\end{bmatrix}
\]

This will get divided by 2, so we can't just keep 2 unchanged!

\[z' = f + f - \frac{f}{z}\]

If \(z = d\), \(z' = f + f - \frac{f}{d}\) unchanged

If \(z = 0\), \(z' = f + f - \frac{f}{0}\) unchanged

At this point, the view volume looks just like it did before orthographic projection! Rays parallel, but wrong dimensions.

So re-use the orth matrix:

\[M_{\text{proj}} = M_{\text{orth}} \cdot M_{\text{persp}}\]