Lecture 10: Barycentric Coordinates and Ray-Triangle Intersection

We'd like to intersect rays with triangles.

[picture of ray hitting 3D triangle]

One way to do this is to intersect with the plane, then determine whether your position on the plane is within the triangle. We're going to do it this way, but the math will end up solving both in one go.

Ray-Plane Intersection

To intersect a ray with a plane, we need an equation for the plane; ideally it's an implicit equation. Suppose we have a plane that goes through a point **a** and has normal **n**. (see whiteboard notes) for any 3D point **p**, the point lies on the plane if and only if (**p - a**) dot **n** equals zero. In fact, (**p - a**) dot **n** gives the signed distance between **p** and the plane, with positive distances on the same side as the normal and negative on the opposite side.

If we have a triangle **a, b, c**and we want this representation, we just need to calculate the normal. We've done this before: we used **n** = (**b-a**) x (**c - a**). For the point on the plane, we'll just use **a** (we could use any point).

To intersect a ray with a plane, we can plug a point on our parametric ray into the implicit equation for the plane and solve:

$$(\mathbf{p} + t\mathbf{d} - \mathbf{a}) \cdot ((\mathbf{b} - \mathbf{a}) \times (\mathbf{c} - \mathbf{a}) = 0$$

The only unknown here is t, that can be solved for, then plugged back into the ray equation to get the 3D coordinates of the intersection. Now all we need is to determine whether this point is inside the triangle. There are many ways to do this, but we're going to do it using **barycentric coordinates**. This is going to seem weird and unnecessary but it's going to end up giving us what we need to check whether the intersection point is inside the triangle or not.

If that were it, this would be unnecessarily complex, but these coordinates are also really convenient for other purposes. Remember that once we hit a point, we need to know surface properties like color, texture coordinates, normals, etc. On triangle meshes these are usually interpolated from vertex data stored at the triangle's corners. Expressing your position on the plane in barycentric coordinates makes for a very elegant answer to "how much do I weight the value from each vertex?"

See whiteboard notes for the presentation of barycentric coordinates. Also see section 2.7 of the book.

Super Cool Properties of Barycentric Coordinates

- $\alpha + \beta + \gamma = 1$, no matter where you are in the plane.
- Each coordinate is the *scaled signed distance* to one of the triangle's edges. That is, the coordinate is 0 at the edge and 1 at the vertex opposite the edge. In particular:
    - α is the fraction of the distance from edge **bc** to vertex **a**
    - β is the fraction of the distance from edge **ac** to vertex **b**
    - γ is the fraction of the distance from edge **ab** to vertex **c**
- (α, β, γ) is inside the triangle iff:

$$\begin{gathered} 0 < \alpha < 1 \\ 0 < \beta < 1 \\ 0 < \gamma < 1 \end{gathered}$$

- The coordinates are proportional to the areas of the subtriangles made by **p** inside the triangle, as a fraction of the full triangle's area, are proportional to the coordinates - see the picture on the slides.

These coordinates, if we find them, solve ray-triangle intersection using the "inside the triangle" property above.

Moreover, they also solve the vertex-data interpolation problem! The fact that they sum to 1 and are proportional to subtriangle areas also makes them perfect for smoothly interpolating vertex ata. If we have some values (e.g., texture coordinates, color values, ...), call them v_a, v_b, v_c, stored at each vertex, we can weight them by their barycentric coordinates to interpolate smoothly anywhere in the triangle:

$$v = \alpha v_a + \beta v_b + \gamma v_c$$

Barycentric Ray-Triangle Intersection

A point in the triangle's plane is $\mathbf{a} + \beta(\mathbf{b} - \mathbf{a}) + \gamma(\mathbf{c} - \mathbf{a})$, and a point on the ray is $\mathbf{p} + t\mathbf{d}$, so we can set these equal to find the intersection point:

$$\mathbf{p} + t\mathbf{d} = \mathbf{a} + \beta(\mathbf{b} - \mathbf{a}) + \gamma(\mathbf{c} - \mathbf{a})$$

The slides show the steps to rearrange these into a 3x3 linear system of the form $\mathbf{Ax} = \mathbf{b}$, which can be solved in a number of ways, but the (4.4.2) shows how to use Cramer's rule to solve it using the fewest possible operations.