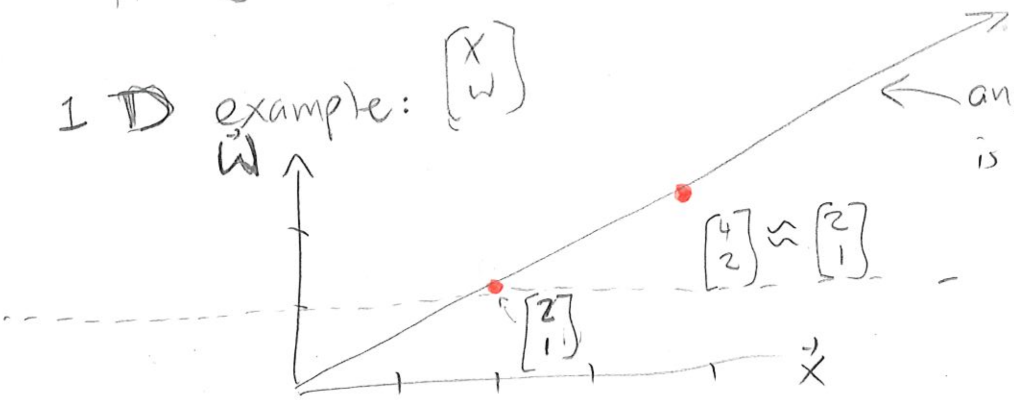
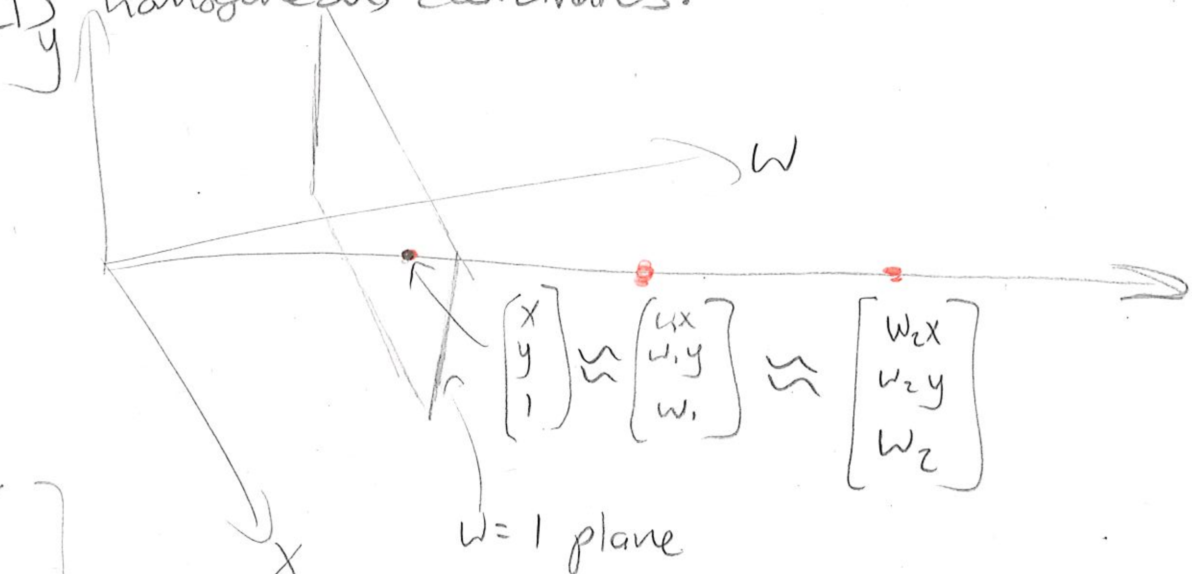
Homogeneous Coordinates: Not 3D, but let's pretend.

1D example: $\begin{bmatrix} X \\ W \end{bmatrix}$



← anything on this "line" is $\backsimeq \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

In 2D homogeneous coordinates:



$$\begin{bmatrix} x \\ y \\ 1 \end{bmatrix} \backsimeq \begin{bmatrix} w_1 x \\ w_1 y \\ w_1 \end{bmatrix} \backsimeq \begin{bmatrix} w_2 x \\ w_2 y \\ w_2 \end{bmatrix}$$

w = 1 plane

"projective plane"

Affine:

$$\begin{bmatrix} a & b & c \\ d & e & f \\ \boxed{0 \quad 0 \quad 1} \end{bmatrix} \begin{bmatrix} \\ \\ 1 \end{bmatrix}$$

points can't leave projective plane

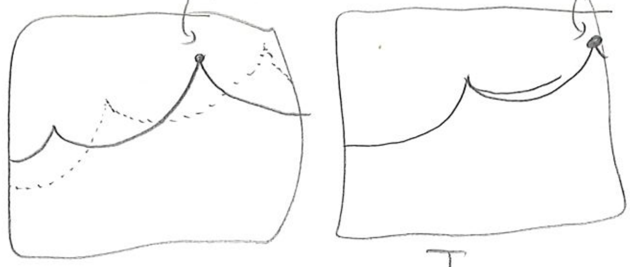
Homography

$$\begin{bmatrix} & & \\ & & \\ g & h & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

can get

Translation

Single Match



matches

unknowns ①

$$x_i' = x_i + x_t$$
$$y_i' = y_i + y_t$$

$$x_t = x_i' - x_i$$
$$y_t = y_i' - y_i$$

Transformation Fitting
- T_x
- Affine
- Homography
  ↳ SVD
  $\|Ah = 0\|$

More Matches: Average them I guess?

$$x_t = \frac{1}{n}\sum_{i=0}^{n}(x_i' - x_i)$$
$$y_t = \frac{1}{n}\sum_{i=0}^{n}(y_i' - y_i)$$

same unknown

$$x_t = x_1' - x_1$$
$$x_t = x_2' - x_2$$
$$\vdots$$

More Matches: Linear Algebra Edition

$$x_t = x_1' - x_1 \qquad y_t = y_1' - y_1$$
$$\vdots \qquad\qquad\qquad \vdots$$

$$r_{x_i}(x_t) = x_i' - (x_i + x_t)$$
$$r_{y_i}(y_t) = y_i' - (y_i + y_t)$$

2n equations
2 unknowns.

Least Squares!

$$\text{Min } \|Ax - b\|^2$$

← residuals

unknowns

$$\underset{A}{\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \\ 0 & 1 \\ & \vdots & \\ 1 & 0 \\ 0 & 1 \end{bmatrix}} \underset{t}{\begin{bmatrix} x_t \\ y_t \end{bmatrix}} - \underset{b}{\begin{bmatrix} x_1' - x_1 \\ y_1' - y_1 \\ x_2' - x_2 \\ \vdots \\ \vdots \\ x_n' - x_n \\ y_n' - y_n \end{bmatrix}}$$

Affine

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} a & b & c \\ d & e & f \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

Unknowns: a, b, c, d, e, f (6)

Eqns: 2 per match

$$x_i' = a x_i + b y_i + c$$

$$y_i' = d x_i + e y_i + f$$

Residuals:

$$(a x_i + b y_i + c) - x_i'$$

$$(d x_i + e y_i + f) - y_i'$$

$$\text{Min} \; \| A \underset{\uparrow \text{unk}}{t} - b \|^2$$

$$\begin{bmatrix} x_1 & y_1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & x_1 & y_1 & 1 \\ x_2 & y_2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & x_1 & y_1 & 1 \\ & & \vdots & & & \\ x_n & y_n & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & x_n & y_n & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \\ e \\ f \end{bmatrix} - \begin{bmatrix} x_1' \\ y_1' \\ x_2' \\ y_2' \\ \vdots \\ x_n' \\ y_n' \end{bmatrix}$$

Homography

$$\begin{bmatrix} x_h \\ y_h \\ w_h \end{bmatrix} = \begin{bmatrix} h_{00} & h_{01} & h_{02} \\ h_{10} & h_{11} & h_{12} \\ h_{20} & h_{21} & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

make this h_{22}

then rewrite

$$x' = \frac{\overbrace{h_{00}x + h_{01}y + h_{02}}^{x_h}}{\underbrace{h_{20}x + h_{21}y + 1}_{w_h}}$$

$$y' = \frac{h_{10}x + h_{11}y + h_{12}}{w_h}$$

Not linear!

$$\frac{h_{00}x + h_{01}y + h_{02}}{\boxed{h_{20}x + h_{21}y + h_{22}}} = x'$$

$$\begin{bmatrix} x_h \\ y_h \\ w_h \end{bmatrix} = \begin{pmatrix} h_{00} & h_{01} & h_{02} \\ h_{10} & h_{11} & h_{12} \\ h_{20} & h_{21} & h_{22} \end{pmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

$$h_{00}x + h_{01}y + h_{02} = x'(h_{20}x + h_{21}y + h_{22})$$

X Residual: $(h_{00}x + h_{01}y + h_{02}) - x'(h_{20}x + h_{21}y + h_{22})$

$$= (h_{00}x + h_{01}y + h_{02}) - (h_{20}xx' + h_{21}yx' + h_{22}x')$$

$$\frac{h_{10}x + h_{11}y + h_{12}}{h_{20}x + h_{21}y + h_{22}} = y'$$

y residual: $h_{10}x + h_{11}y + h_{12} - (h_{20}xy' + h_{21}yy' + h_{22}y')$

We had: $x' = \frac{x_h}{w_h}$ now $x_h = x'w_h$

$y' = \frac{y_h}{w_h}$ now $y_h = y'w_h$

Residuals:

$$x: \overbrace{h_{00}x + h_{01}y + h_{02}}^{x_h} - \overbrace{h_{20}xx' - h_{21}x'y - h_{22}x'}^{x'w_h}$$

$$y: h_{10}x + h_{11}y + h_{12} - h_{20}xy' - h_{21}yy' - h_{22}y'$$

$$\min \|Ah - b\|^2$$

$$\begin{bmatrix} x_1 & y_1 & 1 & 0 & 0 & 0 & x_1x_1' & x_1'y_1 & x_1' \\ 0 & 0 & 0 & x_1 & y_1 & 1 & x_1y_1' & y_1y_1' & y_1' \\ & & & & \vdots & & & & \\ x_n & y_n & 1 & 0 & 0 & 0 & x_nx_n' & x_n'y_n & x_n' \\ 0 & 0 & 0 & x_n & y_n & 1 & x_ny_n' & y_ny_n' & y_n' \end{bmatrix} \begin{bmatrix} h_{00} \\ h_{01} \\ h_{02} \\ h_{10} \\ h_{11} \\ h_{12} \\ h_{20} \\ h_{21} \\ h_{22} \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix}$$

Minimizing $\|Ah-0\|^2$

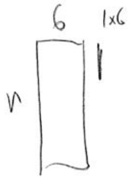
5

Trivial: $h = \vec{0}$

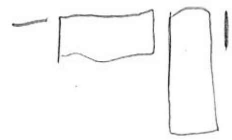
Solution: constrain $\|h\| = 1$

$$\min \|Ah-0\|^2 \quad \text{S.T.} \quad \|h\| = 1$$

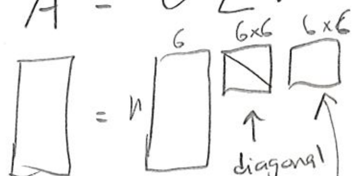
$$\min \|Ah\|^2 \quad \text{S.T.} \quad \|h\| = 1$$



$$\|Ah\|^2 = (Ah)^T(Ah) = h^TA^TAh$$



Singular value decomposition: $A = U\Sigma V^T$



orthogonal, unitary : $u_i^T \cdot u_j = 0$

$u_i^T u_i = 1$

$$h^TA^TAh = h^TV\Sigma U^TU\Sigma V^Th$$
$$= h^TV\Sigma\Sigma V^Th$$

$$\Sigma V^Th : \begin{bmatrix} \sigma_1 & & & \\ & \sigma_2 & & \\ & & \sigma_3 & \\ & & & \ddots \end{bmatrix} \begin{bmatrix} — v_1 — \\ — v_2 — \\ — v_3 — \\ \vdots \end{bmatrix} \begin{bmatrix} h \end{bmatrix}$$

$\|h\| = 1$

$\|v_i\| = 1$

$v_i^T \cdot v_j = 1$

$h = v_3 \rightarrow$

$\sigma_1 v_1 v_3 \rightarrow 0$

$\sigma_2 v_2 v_3 \rightarrow 0$

$\sigma_3 v_3 v_3 \rightarrow \sigma_3$

$\vdots$

0

$$= \begin{bmatrix} \sigma_1 & & & \\ & \sigma_2 & & \\ & & \sigma_3 & \\ & & & \ddots \end{bmatrix} \begin{pmatrix} v_1 h \\ v_2 h \\ v_3 h \\ \vdots \end{pmatrix} = \begin{pmatrix} \sigma_1 v_1 h \\ \sigma_2 v_2 h \\ \sigma_3 v_3 h \\ \vdots \end{pmatrix}$$

6

In Practice:

1. Compute SVD of A.
2. Find index of smallest σ_i in Σ
3. Take the ith column of V (ith row of V^T) as solution h.