CSCI 301 - Lecture 16: More functions

$f(x) = x^2$

Definition: Suppose A, B are sets. A function from A to B, written $f : A \to B$ is a relation ($f \subseteq A \times B$) with the additional property that there is exactly one element $(a, b) \in f$ for each $a \in A$.

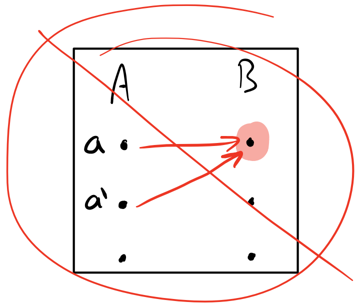
Definitions: If $f : A \to B$ is a function,

- A is the domain of f (The set of possible inputs)
- B is the codomain of f (the set of... things f might map to)
- $\{f(a) : a \in A\}$ is the range of f (The set of... $b \in B$ that f does map to)

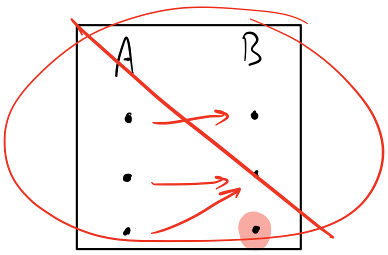
Do Ex. A

Properties of Functions - Injective, Surjective, Bijective

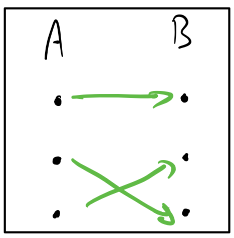
Definitions: A function $f: A \to B$ is



- injective (one-to-one) if
  for all $a, a' \in A$, $a \neq a' \Rightarrow f(a) \neq f(a')$.
  (no two a's map to the same b)



- surjective (onto) if
  for all $b \in B$, there is some $a \in A$ where $f(a) = b$.
  (all b's are mapped to)



- bijective if it is injective and surjective
  (complete matching among a's and b's)

How to show a function $f: A \to B$ is injective: $a \neq a' \Rightarrow f(a) \neq f(a')$

Direct approach:	**Contrapositive approach:**
Suppose $a, a' \in A$ and $a \neq a'$.	Suppose $a, a' \in A$ and $f(a) = f(a')$.
$\vdots$	$\vdots$
Therefore $f(a) \neq f(a')$.	Therefore $a = a'$.

How to show a function $f: A \to B$ is surjective: $\forall b \in B, \exists a \in A, f(a) = b$

Suppose $b \in B$.
[Prove there exists $a \in A$ for which $f(a) = b$.]

$\boxed{\text{Do Ex. B}}$

Inverse Functions

Definition: A function $f: A \to A$ is an identity function if $f(a) = a$ for all a.

Definition: Given a relation R, the inverse relation, written R^{-1}, is defined as
$$\{(y,x) : (x,y) \in R\}$$

Fact: If $f: A \to B$ is a function, then its inverse f^{-1} is a function $f^{-1}: B \to A$ if and only if f is bijective

Fact: If $f: A \to B$ has an inverse, then $f^{-1} \circ f : A \to A$ defined by $\{(a, f^{-1}(f(a)) : a \in A\}$ is an identity function.

$f \circ f^{-1}$ $\qquad f^{-1}(f(x)) = x$

$\boxed{\text{Do Ex. C}}$

Let $a, a' \in \mathbb{Z}$ and $a \neq a'$.

$$f(a) = 2a+1$$
$$f(a') = 2a'+1$$

Suppose $b \in \mathbb{Z}$.

Show $\exists\, a \in A,\ f(a) = b$

$$\longrightarrow 2a+1 = b.$$

Case 1: b is even.

$$\boxed{2a+1 = b}$$

$$f : \underset{A}{\mathbb{Z} \times \mathbb{Z}} \to \underset{B}{\mathbb{Z}},\quad f((m,n)) = 2n - 4m$$

Suppose $a, a' \in A$

$(m,n) \quad (m',n')$

Show: $\exists\, a, a'$ with $a \neq a'$, $f(a) = f(a')$

$m = 2$ $\qquad f((2,1)) = 2 \cdot 2 - 4 \cdot 1 = 0$

$n = 1$ $\qquad f((4,2)) = 2 \cdot 2 - 4 \cdot 1 = 0$

$m' = 4$

$n' = 2$

$2n - 4m = 2(n - 2m)$ $\qquad f(a)$ is even $\forall a$,

so if b is odd, $\nexists\, a$, $f(a) = b$

$f((m,n)) = 3n - 4m$

Injective? No. $\exists\, a, a'$, $a \neq a'$, $f(a) = f(a')$

Let $a = (0,0)$ $\quad f(a) = 0$

$a' = (3,4)$ $\quad f(a') = 3 \cdot 4 - 4 \cdot 3 = 0$ ✓

Surjective? Let $b = 3n - 4m$

$= 3(n-1) - 4(m-1) \leftarrow 3n - 3 - 4m + 4$

$3n - 4m + 1$

Suppose b