

Proposition: Suppose A, B, C are sets, and $C \neq \emptyset$. If $A \times C = B \times C$, then $A = B$.

Proof:

We use direct proof. Suppose A, B, C are sets, $C \neq \emptyset$, and $A \times C = B \times C$.

1. Show $A \times C \subseteq B \times C$

Since $C \neq \emptyset$, there exists some $c \in C$. For any $a \in A$, this means $(a, c) \in A \times C$.

Because $A \times C = B \times C$, we know that $(a, c) \in B \times C$. By definition of the cartesian product, this means that $a \in B$. We have shown that if $a \in A$, then $a \in B$, thus $A \subseteq B$.

2. Show $B \times C \subseteq A \times C$

Since $C \neq \emptyset$, there exists some $c \in C$. For any $b \in B$, this means $(b, c) \in B \times C$.

Because $A \times C = B \times C$, we know that $(b, c) \in A \times C$. By definition of the cartesian product, this means that $b \in A$. We have shown that if $b \in B$, then $b \in A$, thus $B \subseteq A$.