CSCI 301 - Lecture 11 : Proofs Involving Sets

Set Membership

To prove $a \in S$, where

$$S = \{x : P(x)\},$$

Open sentence

we need to show: $P(a)$

$$S = \{x \in A : P(x)\}$$

We need to show:

- $a \in A \wedge P(a)$

Subset Relationships.

Recall: $A \subseteq B$ if: all members of A are also in B.

In other words,

- $\forall x \in A, x \in B$

or equivalently, - $a \in A \Rightarrow a \in B$

Impromptu Ex.: write this symbolically

or equivalently,

$$a \notin B \Rightarrow a \notin A$$

How to Prove $A \subseteq B$ (Direct approach)

Proof. Suppose $a \in A$.
$\vdots$
Therefore $a \in B$.
Thus $a \in A$ implies $a \in B$,
so it follows that $A \subseteq B$. ■

How to Prove $A \subseteq B$ (Contrapositive approach)

Proof. Suppose $a \notin B$.
$\vdots$
Therefore $a \notin A$.
Thus $a \notin B$ implies $a \notin A$,
so it follows that $A \subseteq B$. ■

Set Equality

How can we prove two sets A, B are equal? $A \overset{?}{=} B$

Recall: Two sets are equal if they contain exactly the same elements.

Impromptu Ex.: write this symbolically

In other words,

- $x \in A \Leftrightarrow x \in B$

or equivalently,

- $x \in A \Rightarrow x \in B \;\wedge\; x \in B \Rightarrow x \in A$

$\updownarrow$ $\updownarrow$

$A \subseteq B$ $B \subseteq A$

Notice:

Example

Proposition: Suppose A, B, C are sets and $C \neq \emptyset$.
If $A \times C = B \times C$, then $A = B$.

Proof:

(set up)

Suppose A, B, C are sets and $C \neq \emptyset$.
Further suppose $A \times C = B \times C$.
1. Show: $A \times C \subseteq B \times C$
2. Show: $B \times C \subseteq A \times C$

(type it?)

$\boxed{\text{Do Ex. B}}$

Suppose $A \times C = B \times C$.

First, show $A \subseteq B$.

Since $C \neq \emptyset$, there exists some $c \in C$.
For any $a \in A$, this means $(a, c) \in A \times C$.
Because $A \times C = B \times C$, this means $(a, c) \in B \times C$.
By definition of cartesian product, this means
$a \in B$. Therefore $A \subseteq B$.

Next, show $B \subseteq A$
(proof is similar - see typed notes)

Suppose $x \in \{12n : n \in \mathbb{Z}\}$

$$x = 12n$$
$$= 2(6n) \in \{2n : n \in \mathbb{Z}\}$$
$$= 3(4n) \in \{4n : n \in \mathbb{Z}\}$$