301 L09 - Proof By Contradiction

To prove P, suppose $\neg P$

Then show that everything is ON FIRE

and nothing makes sense!

Symbolically, $P \equiv (\neg P \Rightarrow (C \wedge \neg C))$

P	$\neg P$	$C \wedge \neg C$	$\neg P \Rightarrow C \wedge \neg C$
T	F	F	T
F	T	F	F

Outline for Proof by Contradiction

Proposition P.

Proof. Suppose $\sim P$.

$\vdots$

Therefore $C \wedge \sim C$. ■

Notice:
not $P \Rightarrow Q$!

Notice:
Any C!

Definition/reminder: A real number x is **rational** if $x = \frac{a}{b}$ for some integers a, b. A number that is not rational is **irrational**.

Classic Example:

Proposition: $\sqrt{2}$ is irrational.

Proof (contradiction): Suppose $\sqrt{2}$ is rational.

Then $\sqrt{2} = \frac{a}{b}$ for some $a, b \in \mathbb{Z}$.

Further suppose, $\frac{a}{b}$ is fully reduced.

Therefore a and b are not both even.

So $\sqrt{2} = \frac{a}{b}$, thus $2 = \frac{a^2}{b^2}$. Rearranging,

we can get $a^2 = 2b^2$, so a^2 is even, so a is even.

Let $a = 2c$ for some $c \in \mathbb{Z}$. Then

$a^2 = 2b^2$

$(2c)^2 = 2b^2$, therefore $4c^2 = 2b^2$, or $2c^2 = b^2$.

So b^2 is even, and thus b is even.

$\rightarrow$ We have arrived at a contradiction. ∎

Proving Conditional Statements by Contradiction

$P \Rightarrow Q$

$\neg(P \Rightarrow Q) \equiv P \wedge \neg Q$

Outline for Proving a Conditional Statement with Contradiction

Proposition If P, then Q.

Proof. Suppose P and $\sim Q$.

⋮

Therefore $C \wedge \sim C$. ■

Example

Proposition: If $\underbrace{a, b \in \mathbb{Z} \text{ and } a \geq 2}_{P}$, then $\underbrace{a \nmid b \text{ or } a \nmid b+1}_{Q}$.

Proof (contradiction):

(setup)

Suppose $a, b \in \mathbb{Z}$ and $a \geq 2$ and $a \mid b$ and $a \mid b+1$

Suppose $\exists n \in \mathbb{Z},\ 4 \mid n^2+2.$

$4 \mid n^2+2$ $\qquad n^2+2 = 4d$

n^2+2 is even.

n is even

$n = 2c,\ c \in \mathbb{Z}$

$n^2+2 = 4c^2+2$

$4 \mid 4c^2+2$

$\boxed{\text{Do Ex. A}}$

$\forall n \in \mathbb{Z},\ 4 \nmid (n^2+2)$

Suppose $\exists n \in \mathbb{Z},\ 4 \mid n^2+2$

$n^2+2 = 4c$ for some $c \in \mathbb{Z}$

n is even

$n = 2d$

$n^2+2 = 4c$

$4d^2+2 = 4c$

$2(2d^2+1) = 4c$

$2d^2+1 = 2c$

odd even