301 L13 - Proof By Contradiction

$$P \equiv (\neg P \Rightarrow (C \wedge \neg C))$$

P	$\neg P$	$C \wedge \neg C$	$\neg P \Rightarrow C \wedge \neg C$
T	F	F	T
F	T	F	F

Outline for Proof by Contradiction

Proposition P.

Proof. Suppose $\sim P$.

$\vdots$

Therefore $C \wedge \sim C$. ■

Definition/reminder: A real number x is **rational** if $x = \frac{a}{b}$ for some integers a, b. A number that is not rational is **irrational**.

Classic Example:

Proposition: $\sqrt{2}$ is irrational.

Proof: Suppose $\sqrt{2}$ is rational.

Then, $\sqrt{2} = \frac{a}{b}$ for some integers a, b.

Suppose that $\frac{a}{b}$ is in fully reduced form.

This means a and b are not both even.

So $2 = \frac{a^2}{b^2}$, and $a^2 = 2b^2$.

This implies that a^2 is even, so a is even.

Let $a = 2c$ for some c.

$a^2 = 2b^2$

$(2c)^2 = 2b^2$

$4c^2 = 2b^2$

$2c^2 = b^2$. This implies that b^2 is even, thus b is even, contradicting the fact that a and b are not both even. ∎

Proving Conditional Statements by Contradiction.

Proposition: $P \Rightarrow Q$

Suppose $\neg(P \Rightarrow Q)$

$\equiv$

Suppose $P \wedge \neg Q$.

Outline for Proving a Conditional Statement with Contradiction

Proposition If P, then Q.	
Proof. Suppose P and $\sim Q$.	
$\vdots$	
Therefore $C \wedge \sim C$.	■

Proposition: If $a, b \in \mathbb{Z}$ and $a \geq 2$, then $a \nmid b$ or $a \nmid b+1$.

Ex#1 Proof:

Suppose n is odd and n^2 is even.

Let $n = 2a+1$. So $n^2 = (2a+1)^2 = 4a^2 + 4a + 1 = 2(2a^2+2a)+1$.

This shows n^2 is odd, a contradiction. ■

Ex#2 setup: Suppose $\exists\, n \in \mathbb{Z},\ 4 \mid (n^2+2)$