CSCI 241

Scott Wehrwein

Single Source Shortest Paths:
Weighted Graphs
Dijkstra's Algorithm - Intuition

Goals

Know what a weighted graph is.

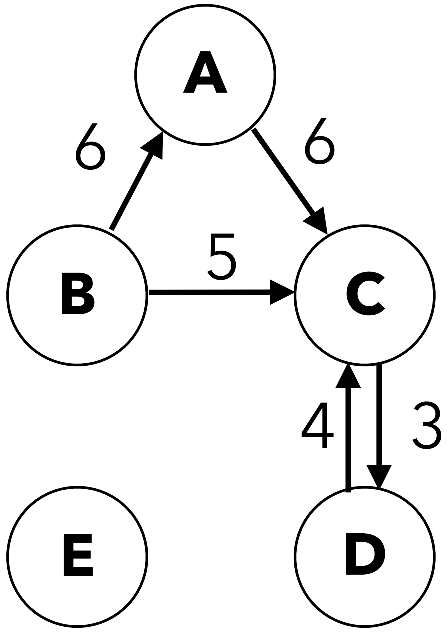
Understand the intuition and high-level pseudocode for Dijkstra’s single-source shortest paths algorithm.

Be able to execute Dijkstra's algorithm on paper.

Weighted Graphs

Like a normal graph, but edges have weights.

Formally: a graph (V,E) with an accompanying *weight function* $w : E \to \mathbb{R}$



- may be directed or undirected.

Informally: label edges with their weights

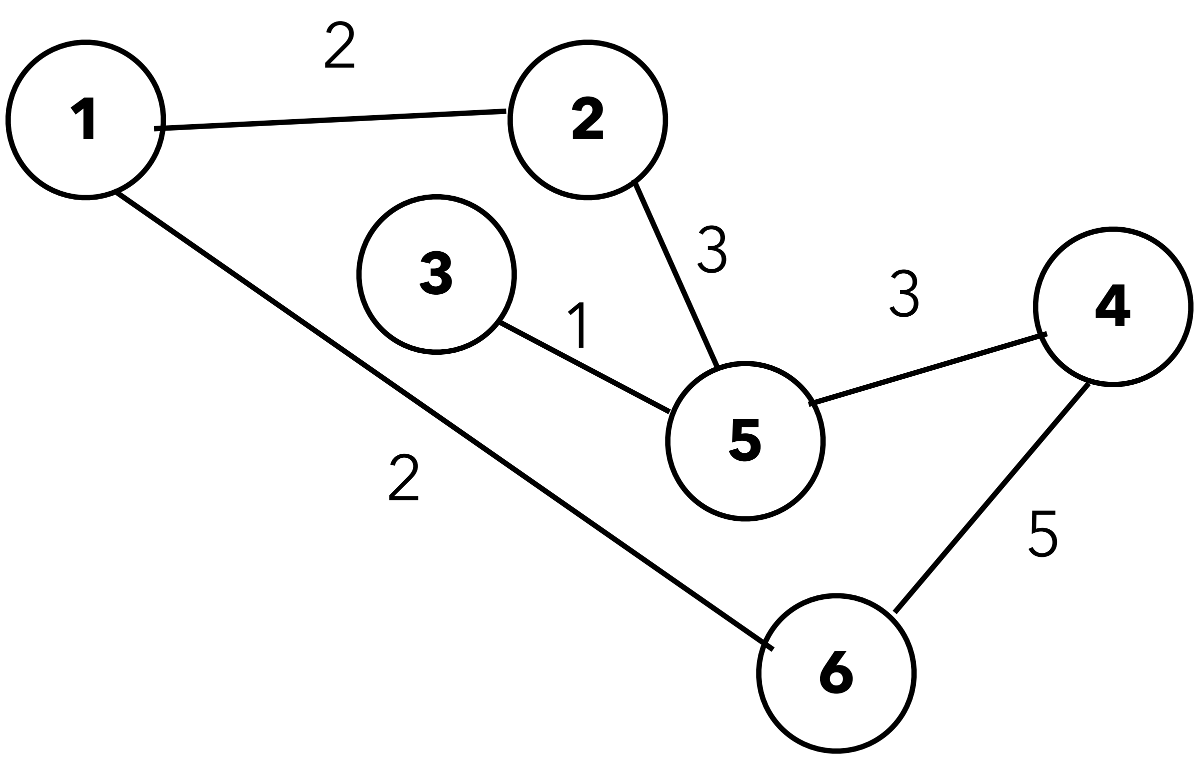
Representation:

- adjacency list - store weight of (u,v) with v the node in u's list
- adjacency matrix - store weight in matrix entry for (u,v)

Paths in Weighted Graphs

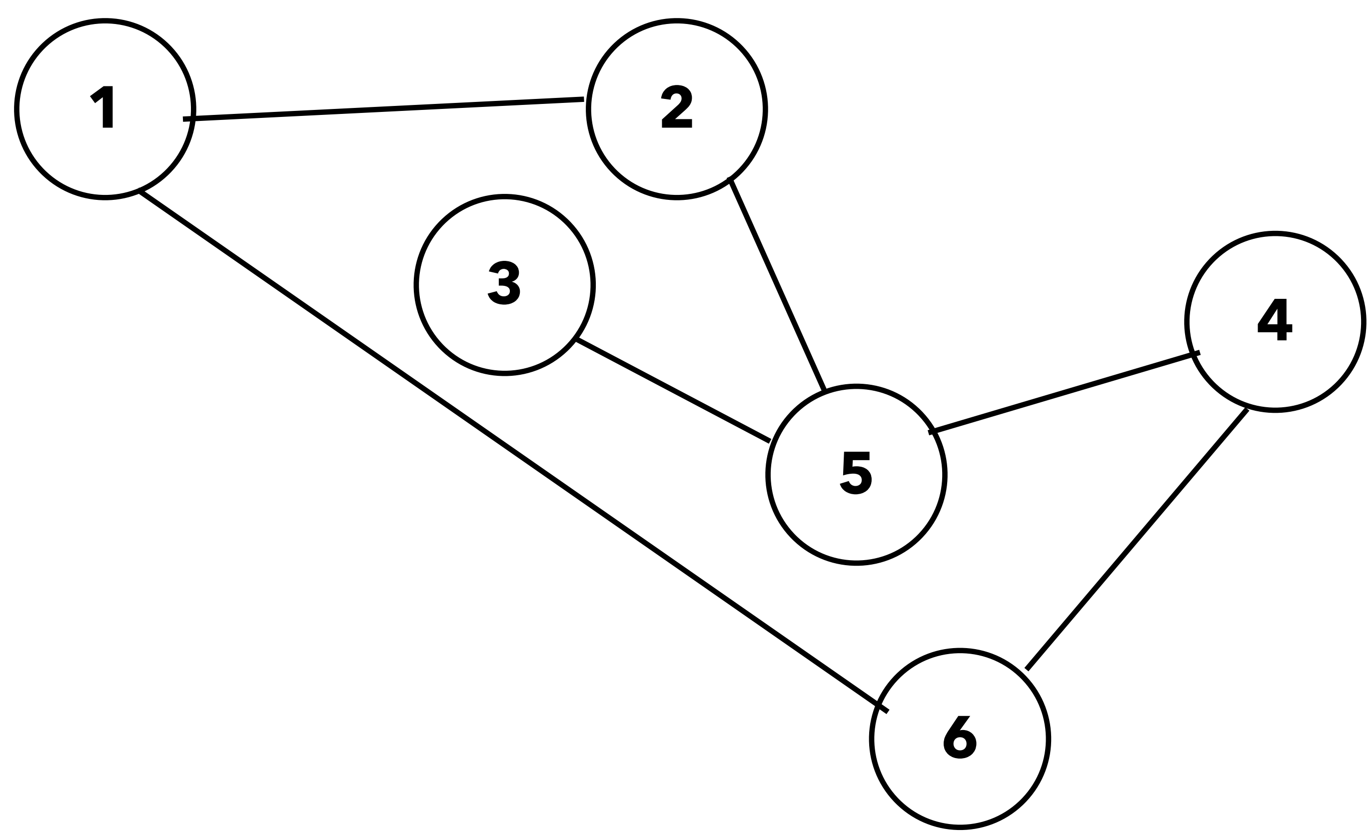
The length (or weight) of a path in a weighted graph is the sum of the edge weights along that path.

Example: the path (1, 6, 4) has weight 7.



Computing Shortest Paths in Unweighted Graphs

- Perform a breadth-first search (that's it!)
- BFS visits nodes in order of "hop distance", or path length!
- BFS(1):



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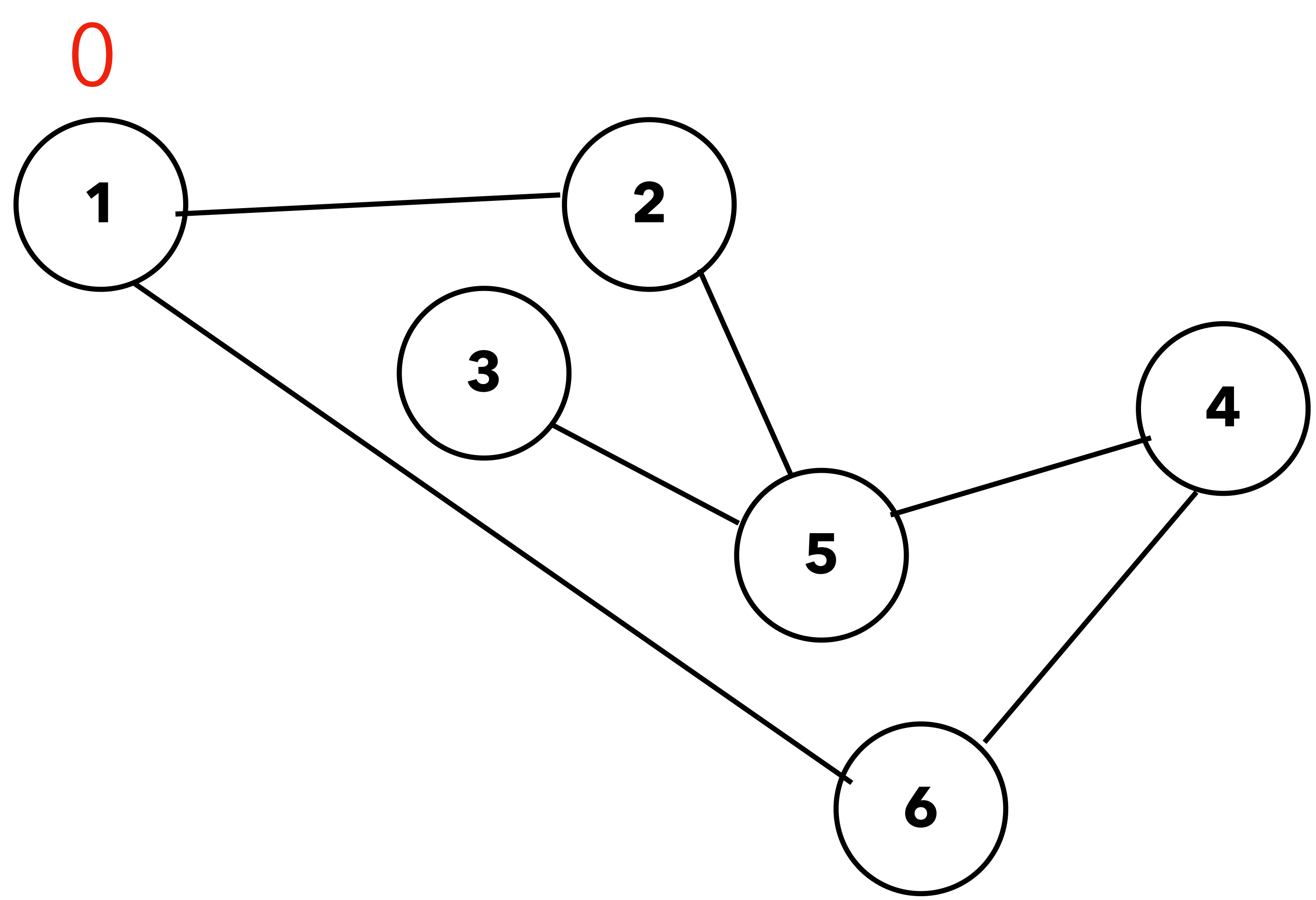

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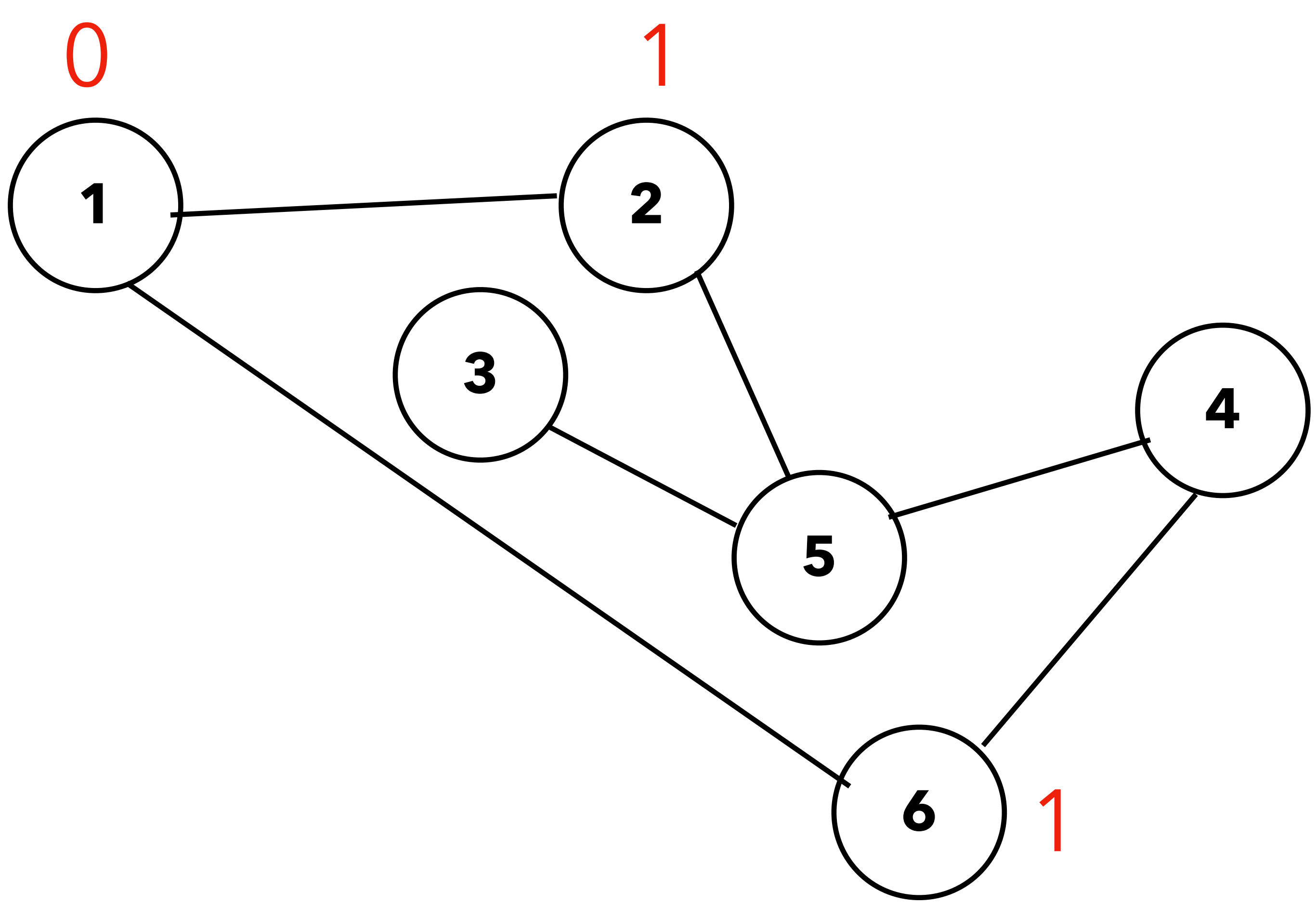

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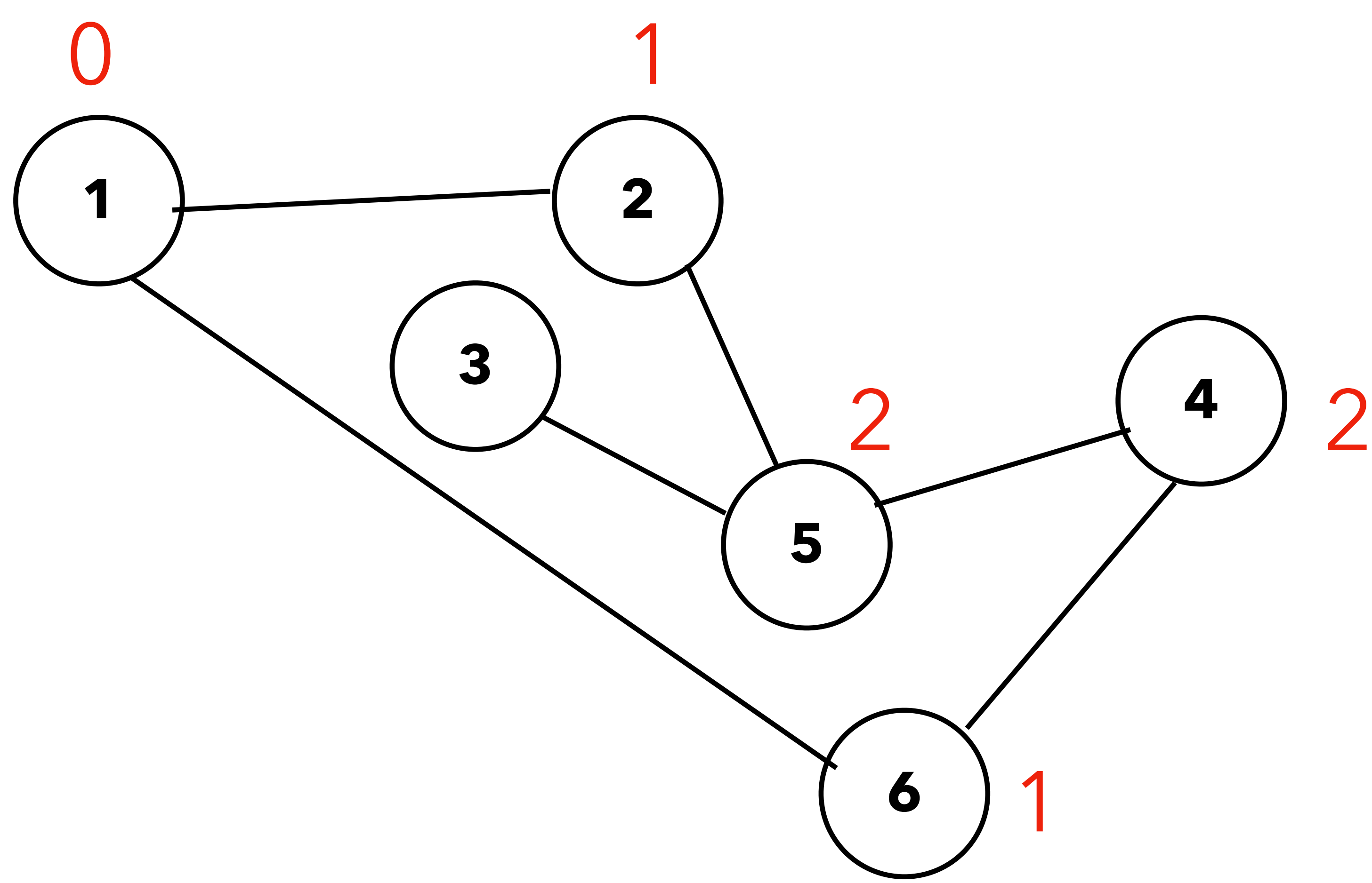

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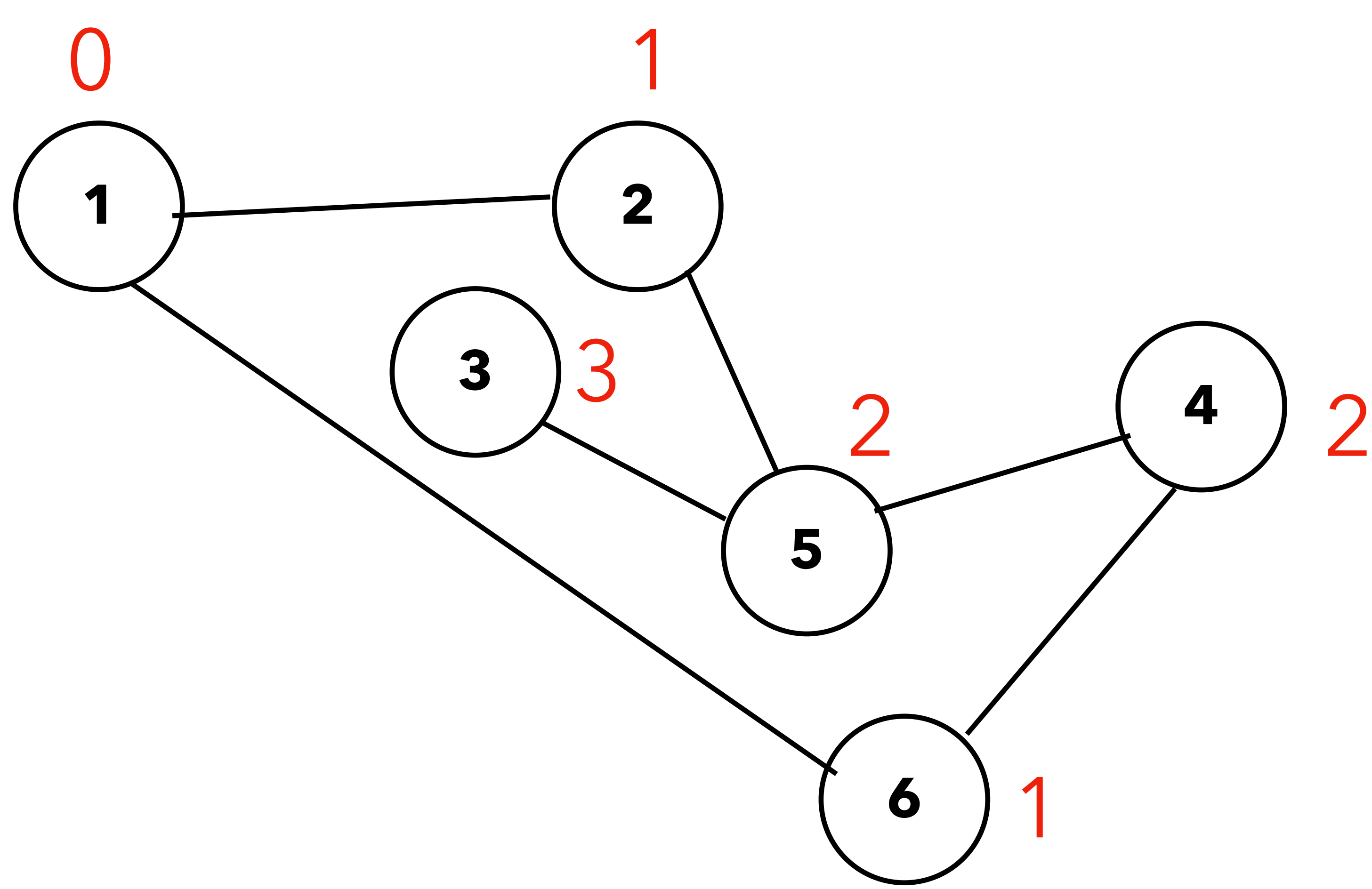
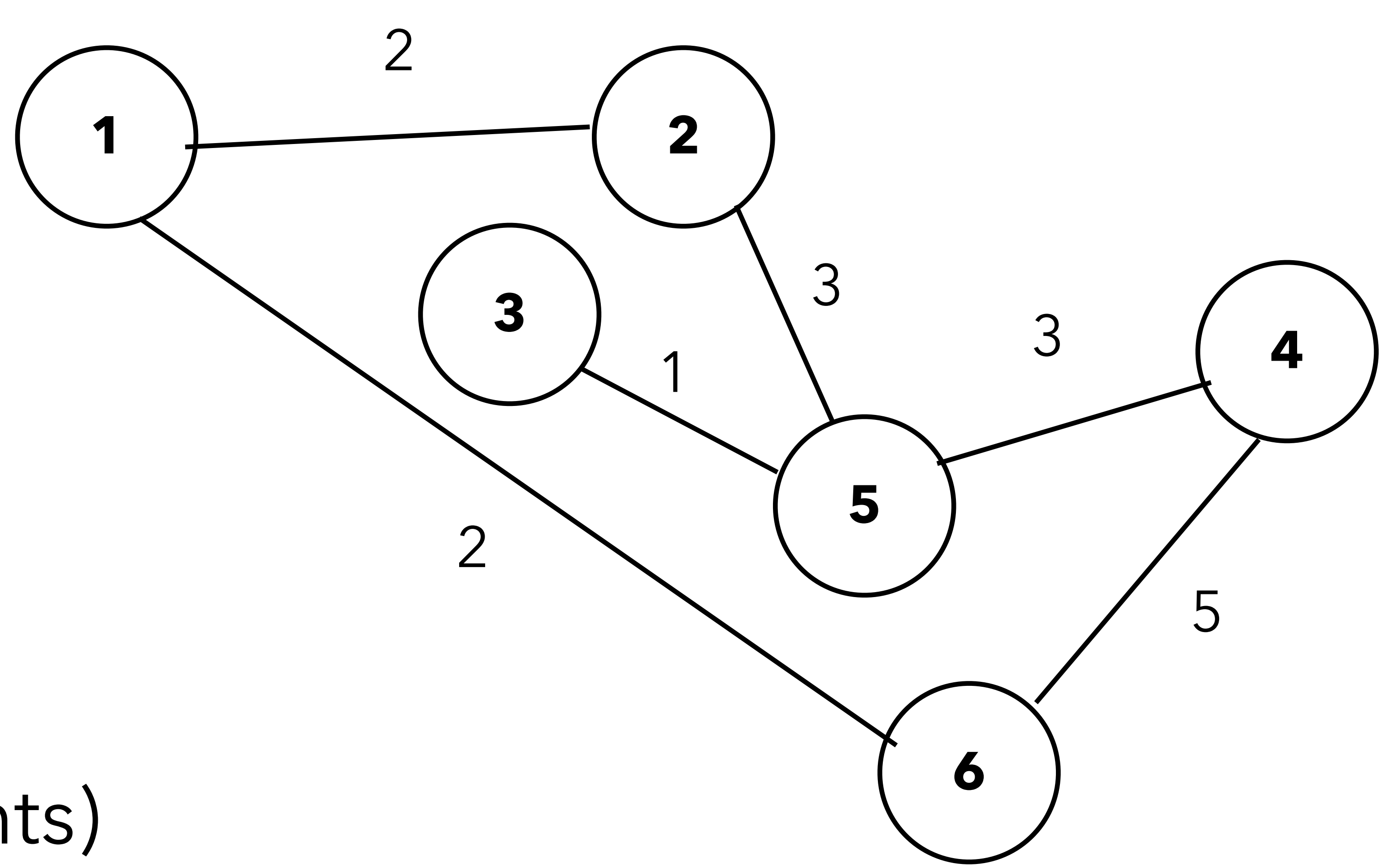


Computing Shortest Paths in Weighted Graphs

BFS doesn't visit nodes in order of shortest path length:



(edge weights)
(shortest path length from node 1)

Computing Shortest Paths in Weighted Graphs

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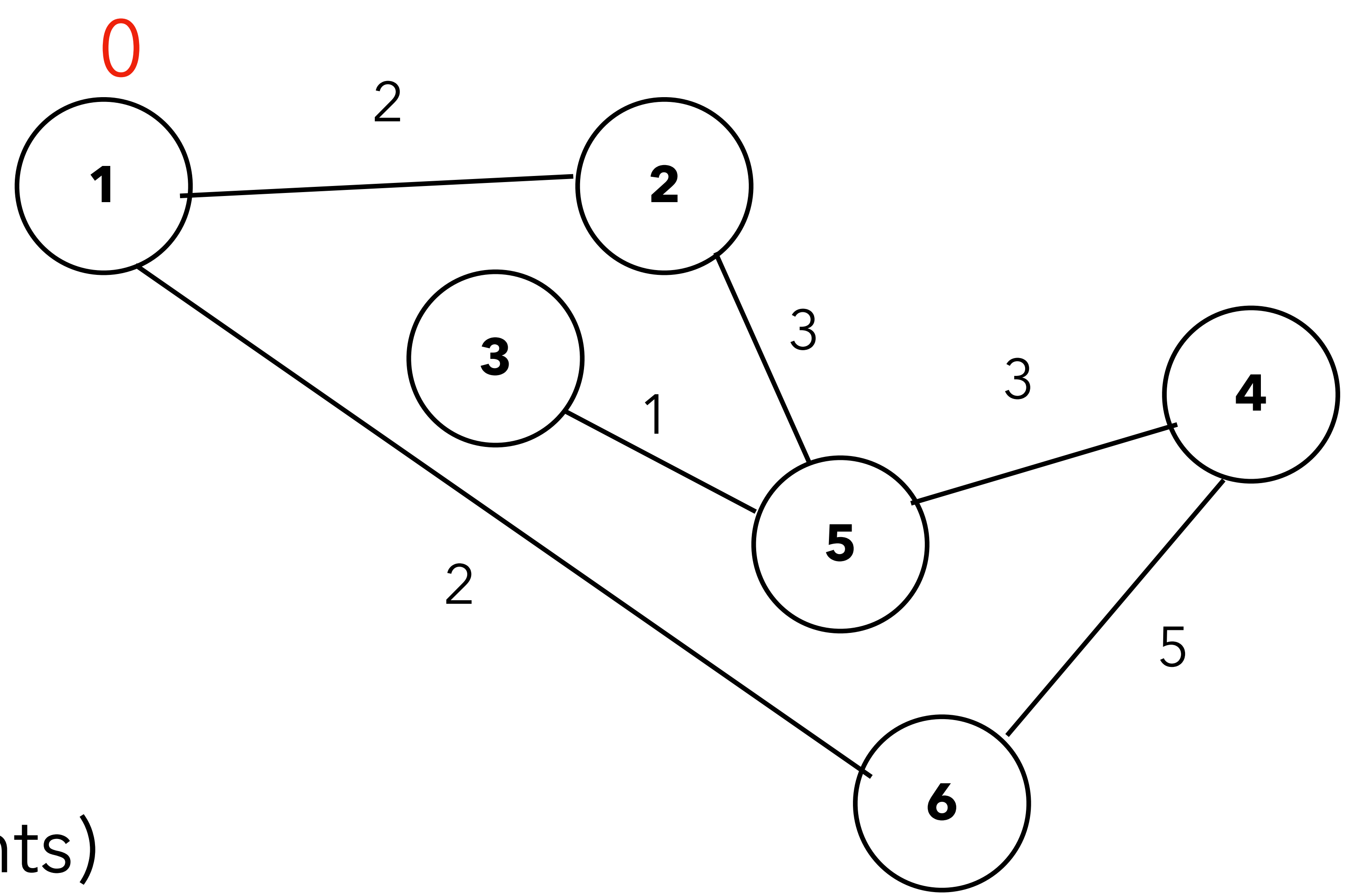

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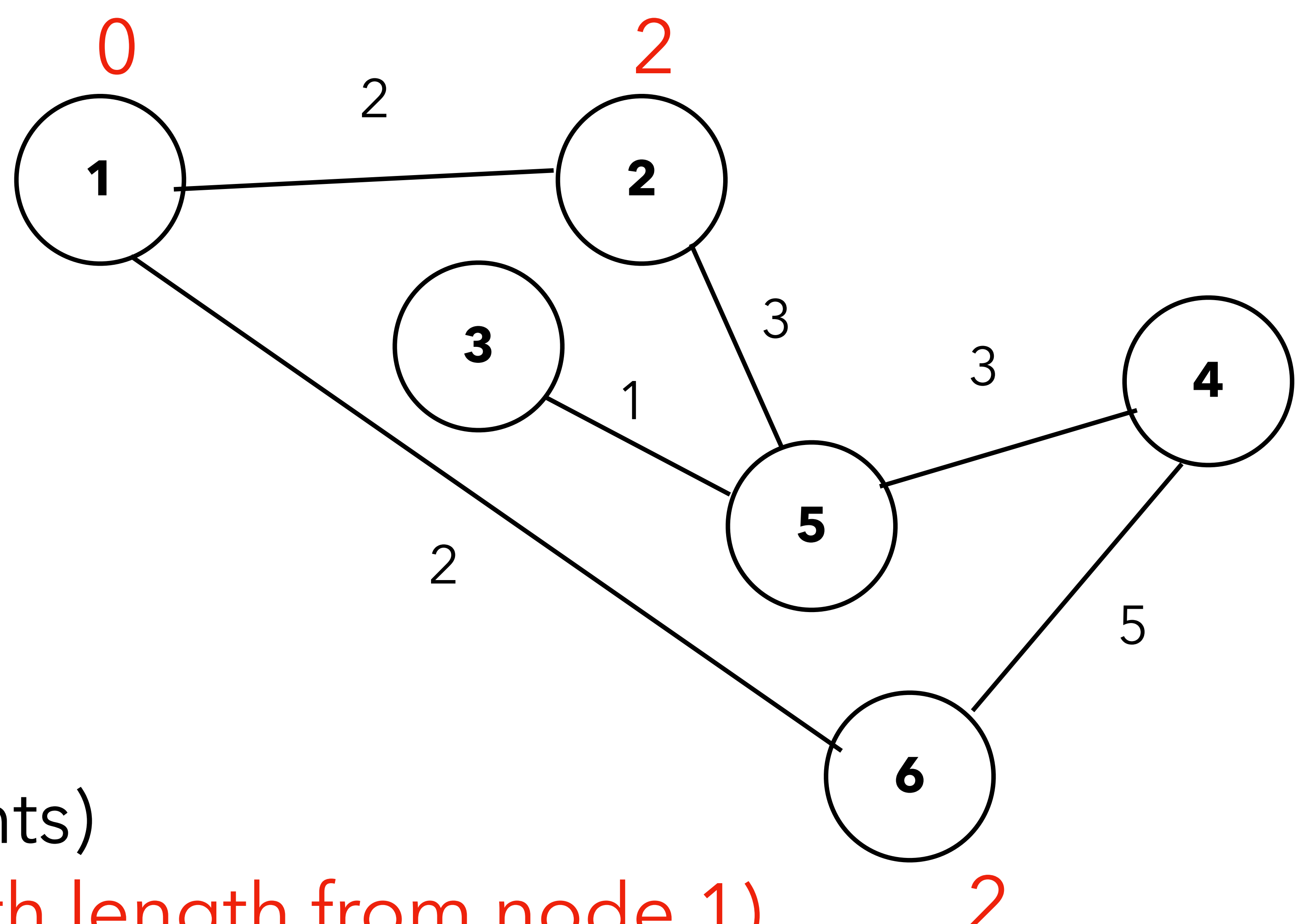

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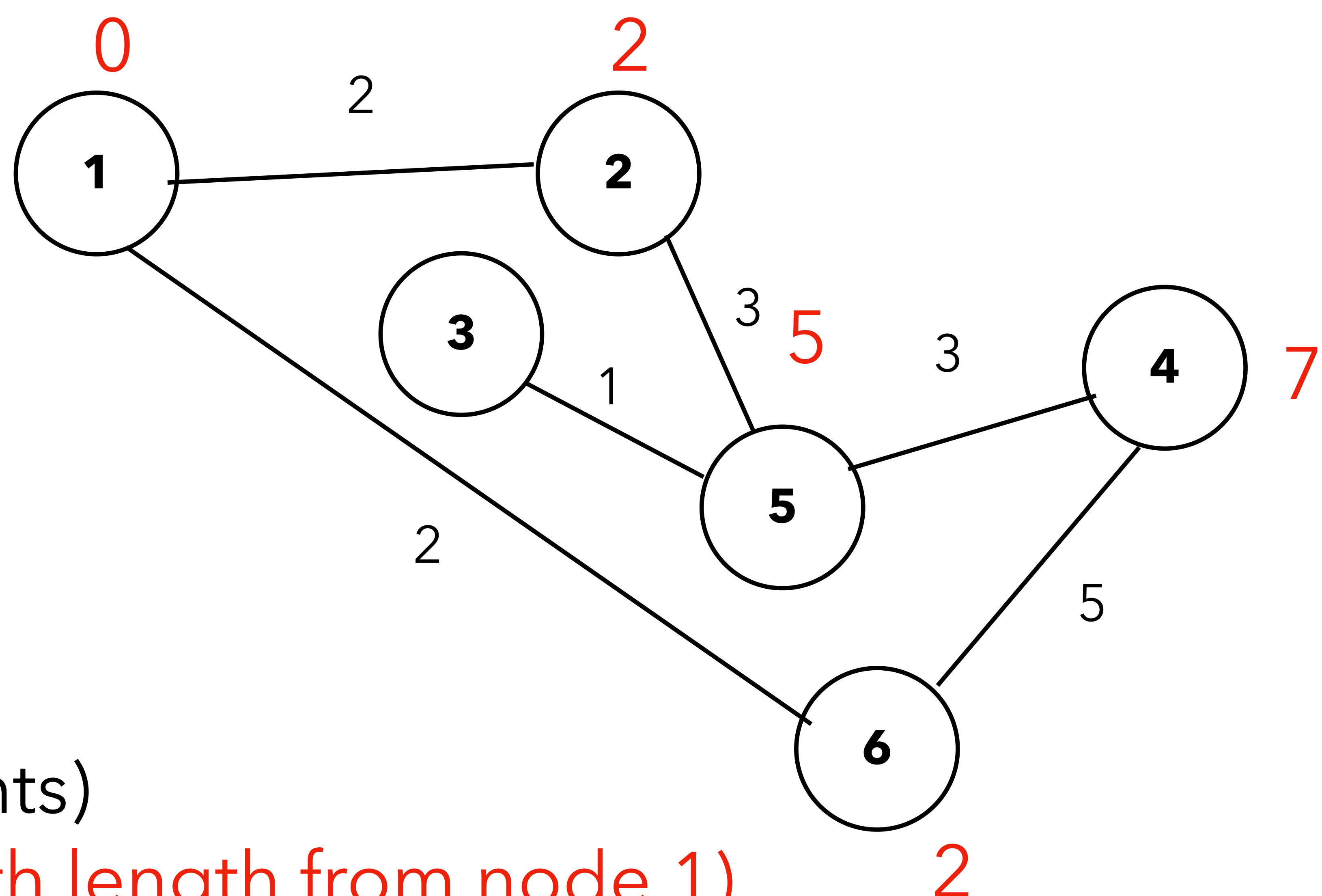

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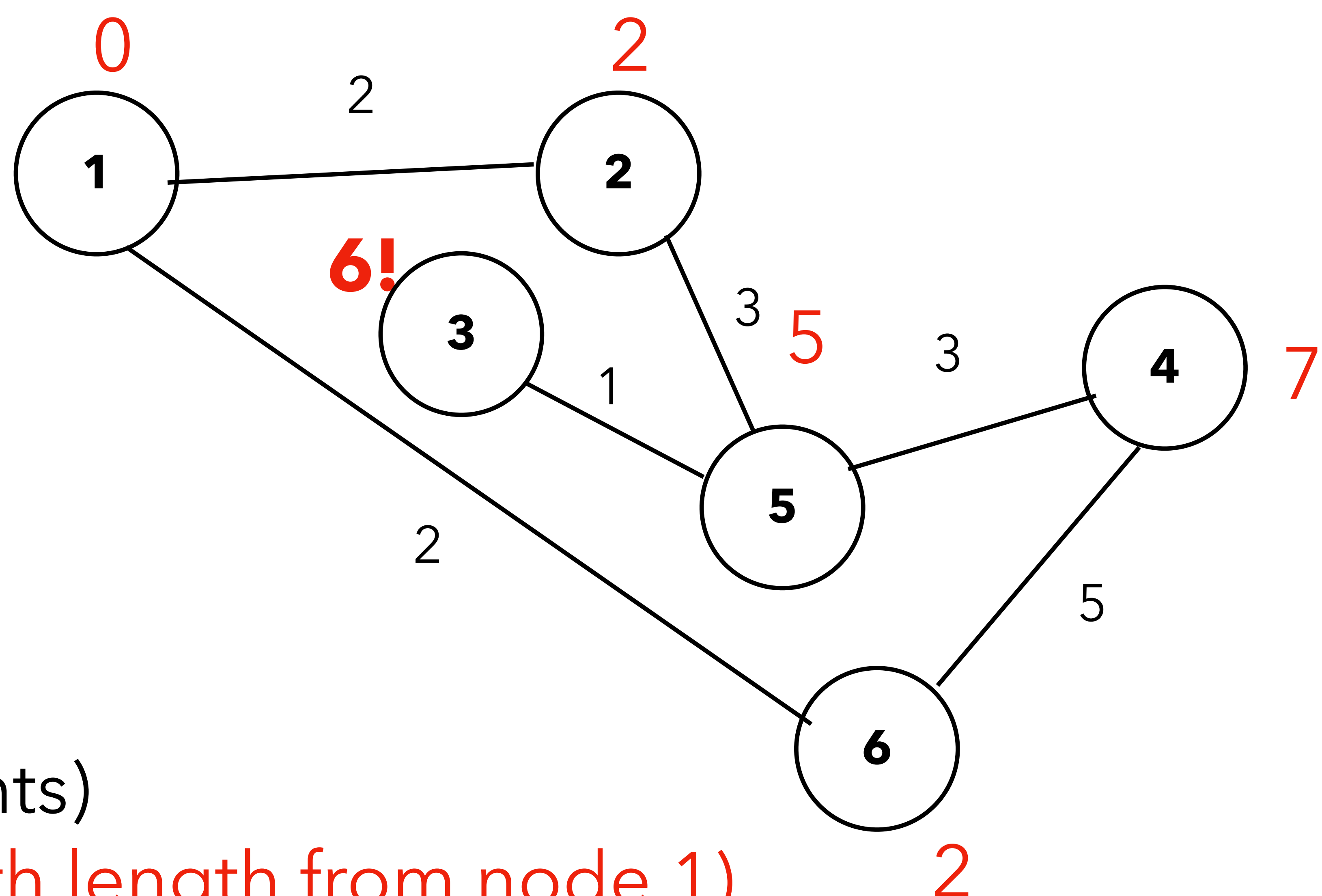

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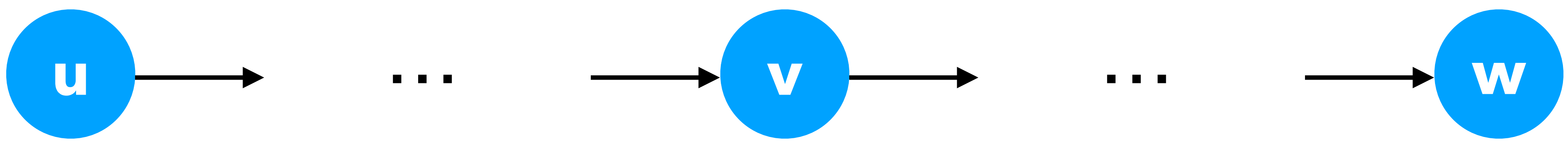
Dijkstra's Shortest Paths: Subpaths

Fact: **subpaths** of shortest paths are shortest paths



Example: if the shortest path from **u** to **w** goes through **v**, then:

- the part of that path from **u** to **v** is the shortest path from **u** to **v**.
- if there were some better path **u**..**v**, that would also be part of a better way to get from **u** to **w**.

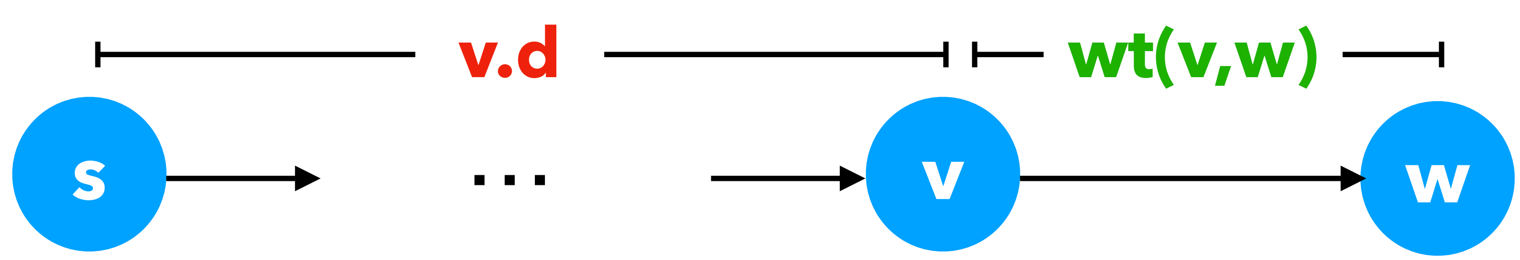
Dijkstra's Shortest Paths: Subpaths

Fact: **subpaths** of shortest paths are shortest paths

Consequence: a **candidate** shortest path from start node **s** to some node **v**'s neighbor **w** is the shortest path from to **v** + the edge weight from **v** to **w**.

Shorthand:

- **v.d** is the shortest (known) distance from the start to **v**
- **wt(v, w)** is the weight of the edge from **v** to **w**



Dijkstra’s Shortest Paths: Intuition

Intuition: **explore nodes like BFS, but in order of path length instead of number of hops.**

There are three kinds of nodes:

- Settled - nodes for which we know the actual shortest path.
- Frontier - nodes that have been visited but we don’t necessarily have their actual shortest path
- Unexplored - all other nodes.

Each node **n** keeps track of **`n.d`**, the length of the shortest known known path from start.

We may discover a shorter path to a frontier node than the one we’ve found already - if so, update **`n.d`**.

Dijkstra's Shortest Paths: Cartoon

settled frontier unexplored

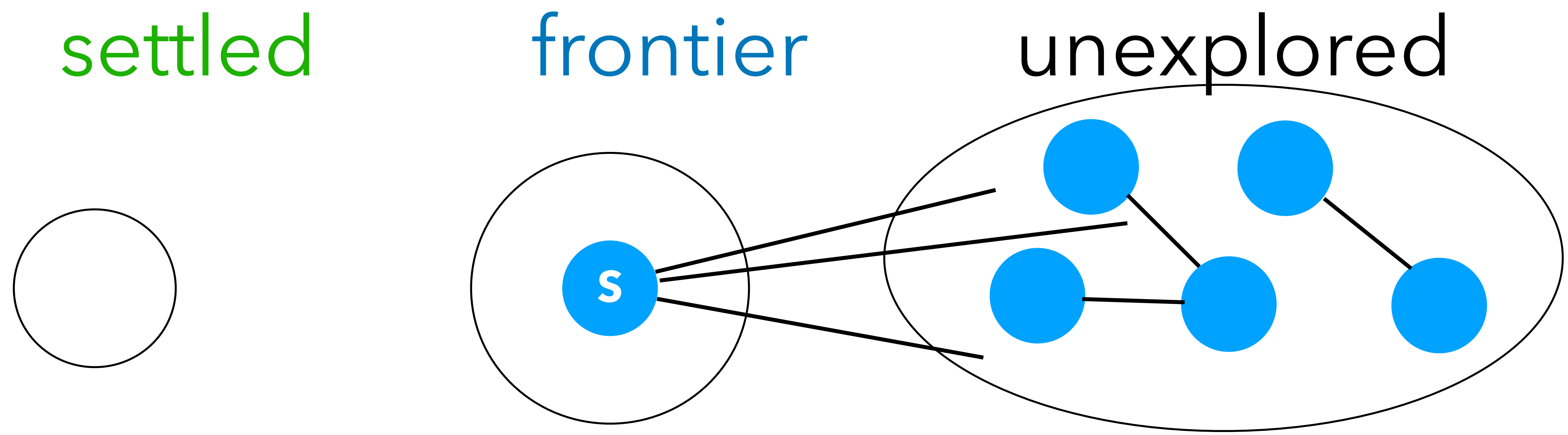
Before:

During:

After:

Dijkstra's Shortest Paths: Cartoon

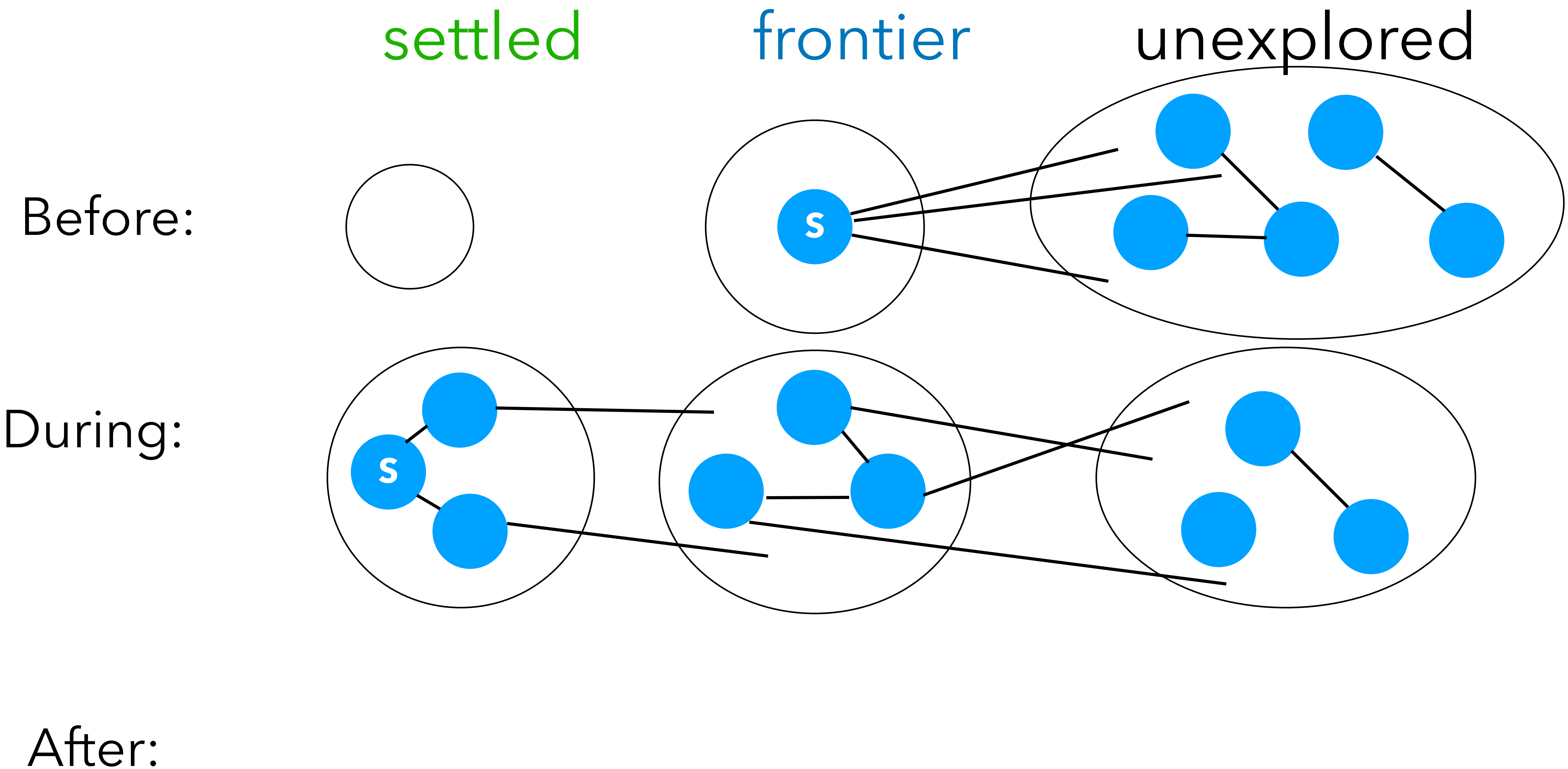
Before:



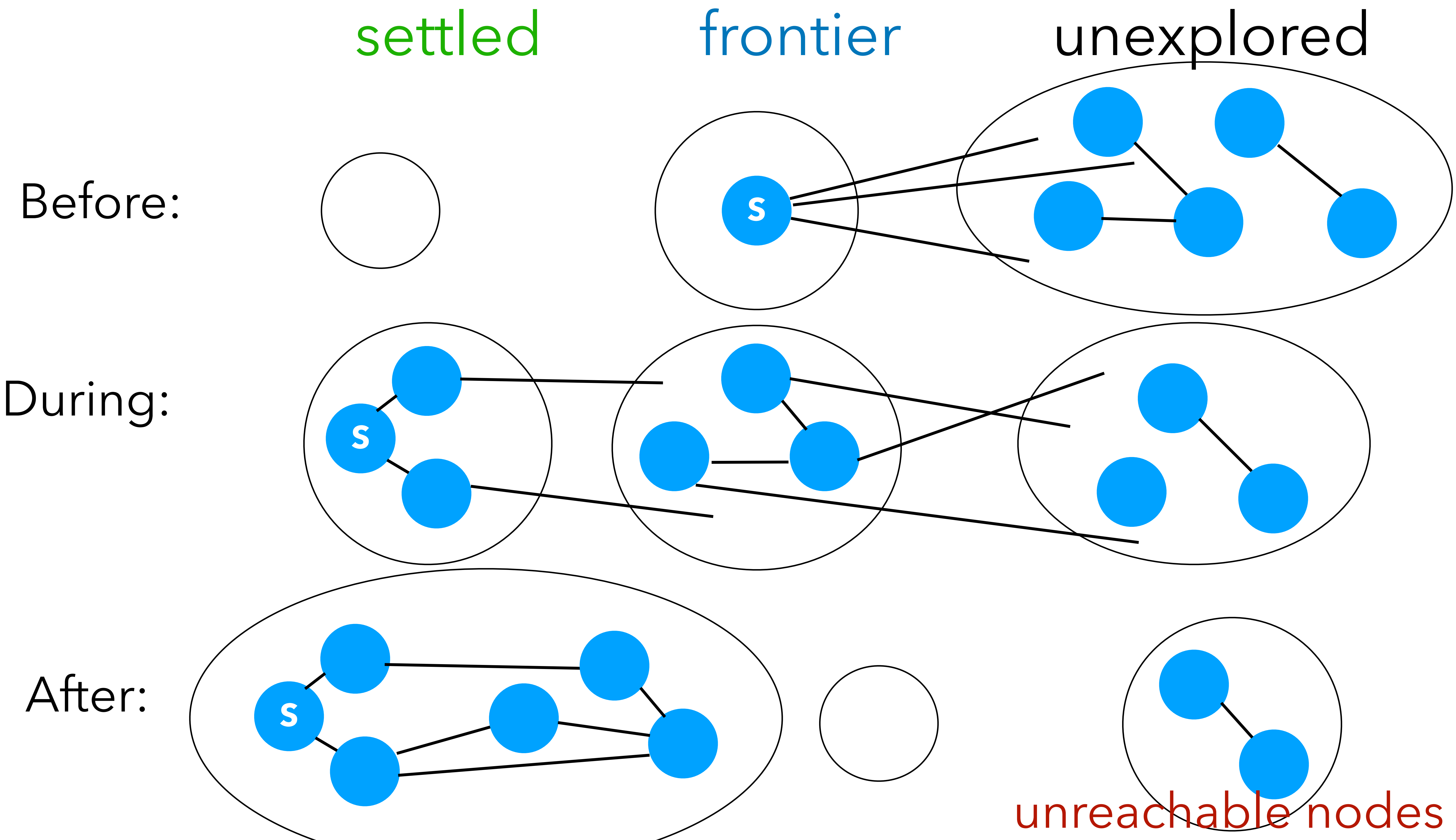
During:

After:

Dijkstra’s Shortest Paths: Cartoon



Dijkstra's Shortest Paths: Cartoon



Dijkstra's Shortest Paths: High-Level Algorithm

```
Initialize Settled to empty
Initialize Frontier to the start node
While the frontier isn't empty:
  move the node f with smallest d from F to S
  For each neighbor w of f:
    if we've never seen w before:
      set its path length
      add it to frontier
    else if the path to w via f is shorter:
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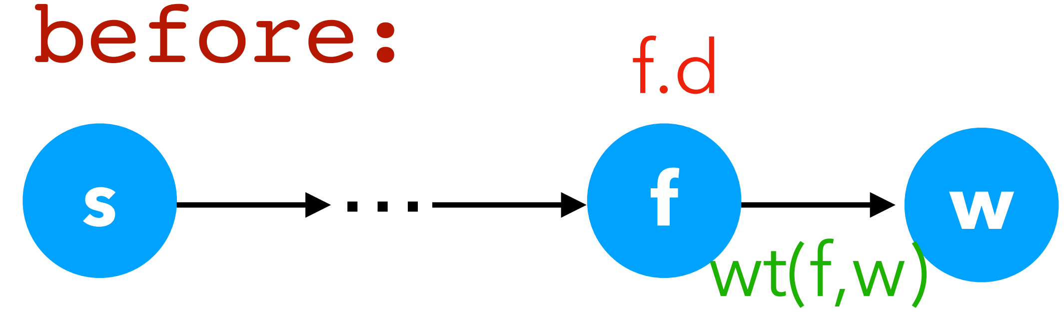

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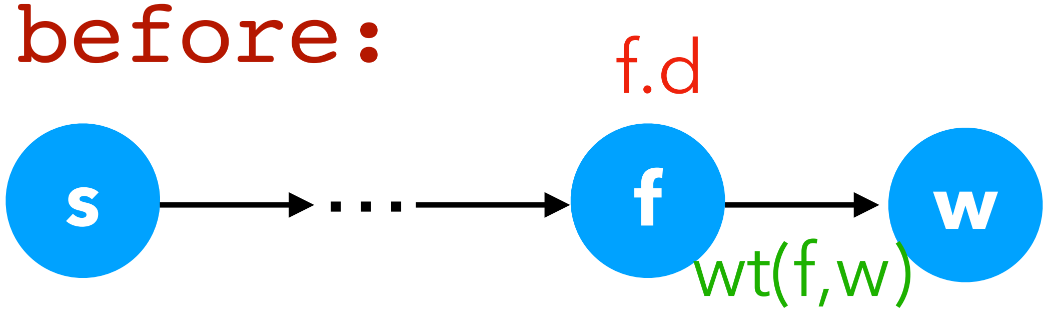

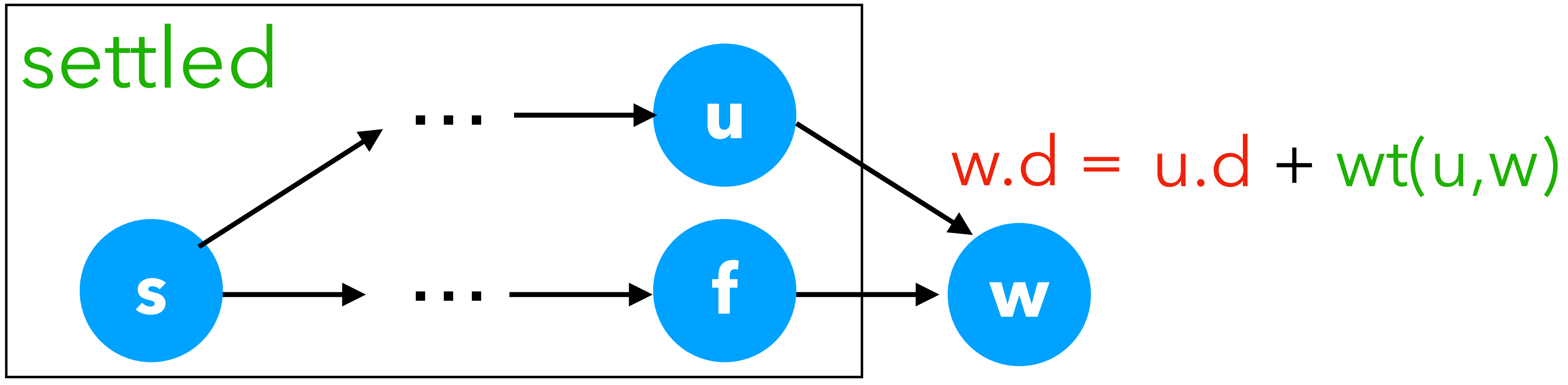

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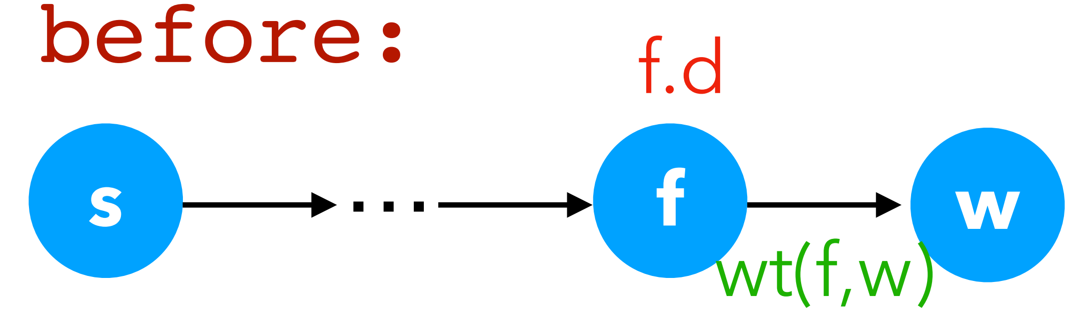

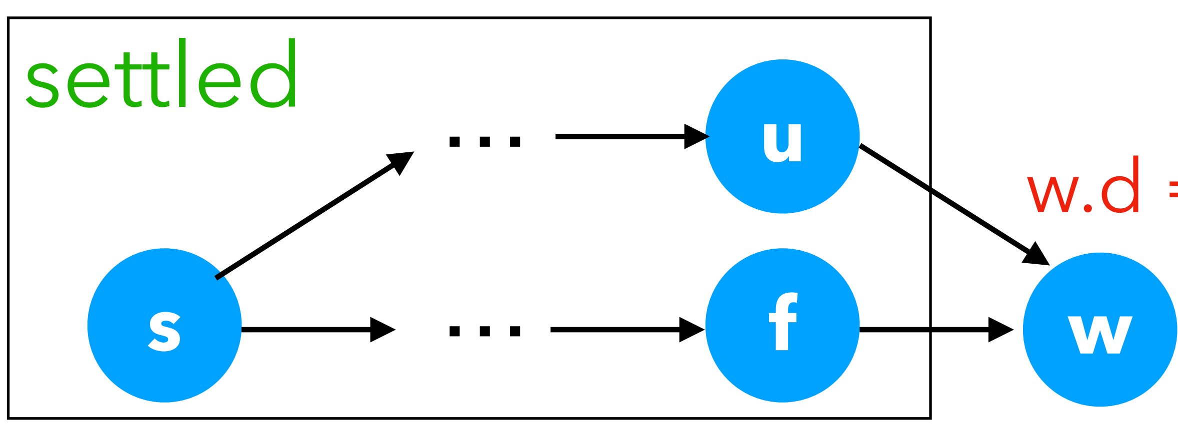

w.d = ~~u.d + wt(u,w)~~

f.d + wt(f,w)

Dijkstra's Shortest Paths: Execution

Best known distances:

Node	d
0	?
1	?
2	?
3	?
4	?

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    if we've never seen w before:
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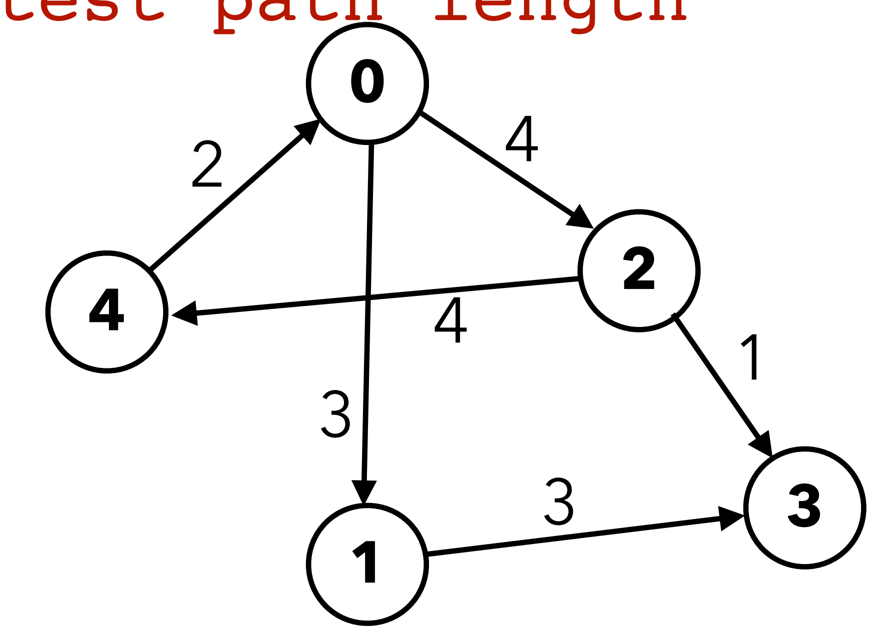
```

Settled set:

Frontier set:



shortest-paths(4)

Dijkstra’s Shortest Paths: Execution

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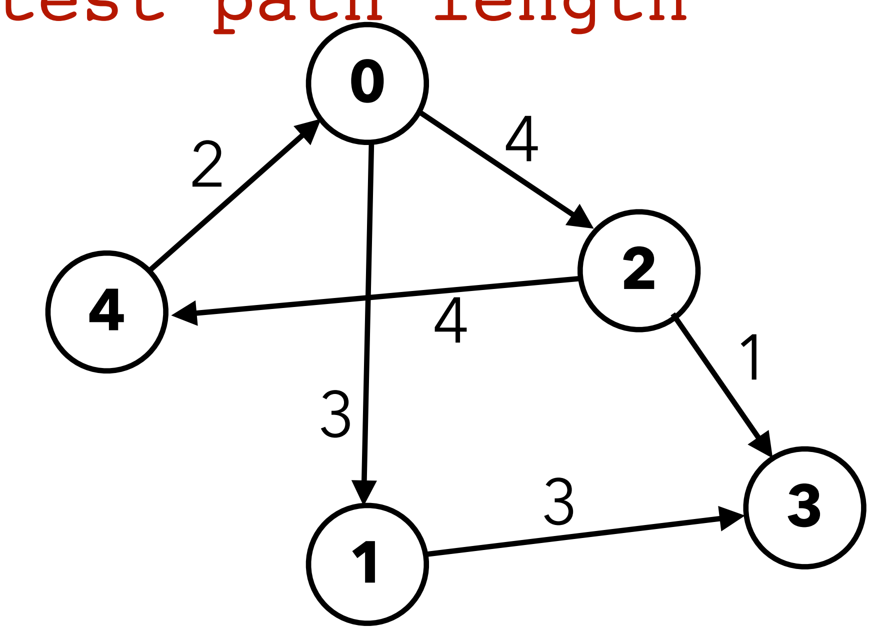
```

Node	d
0	?
1	?
2	?
3	?
4	0

Settled set: {}

Frontier set: {4}



shortest-paths(4)

Dijkstra’s Shortest Paths: Execution

Node	d
0	?
1	?
2	?
3	?
4	0

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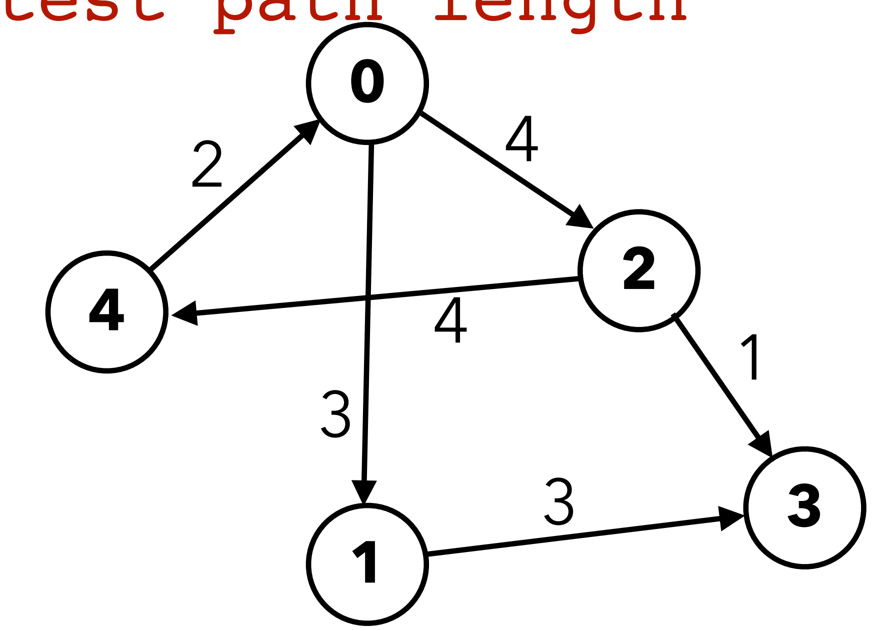
```

f: 4

Settled set: {4}

Frontier set: {}



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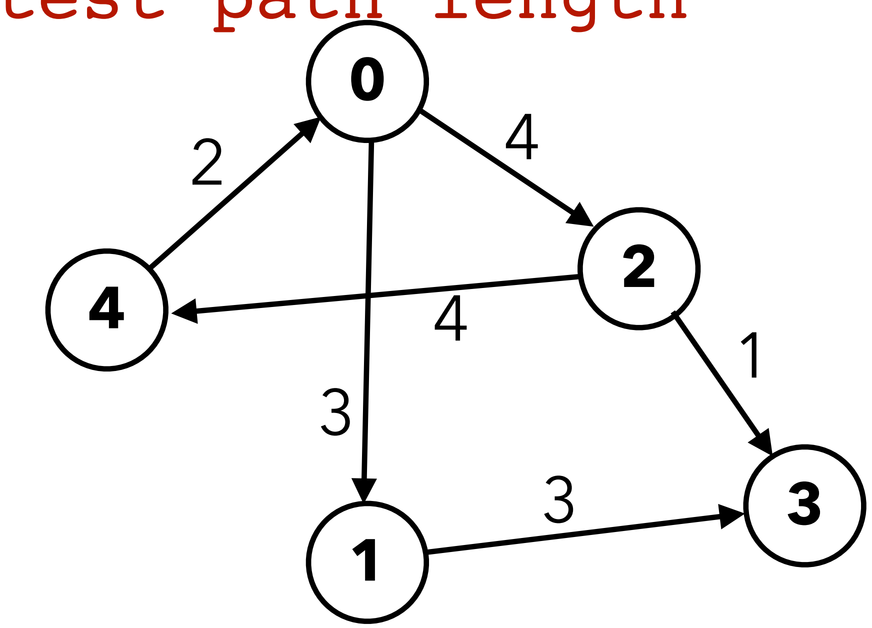
      add w to the frontier
    else if the path to w via f is shorter:
      update w's shortest path length
```

f: 4
w: 0

Node	d
0	**2**
1	?
2	?
3	?
4	0

Settled set: {4}

Frontier set: {0}



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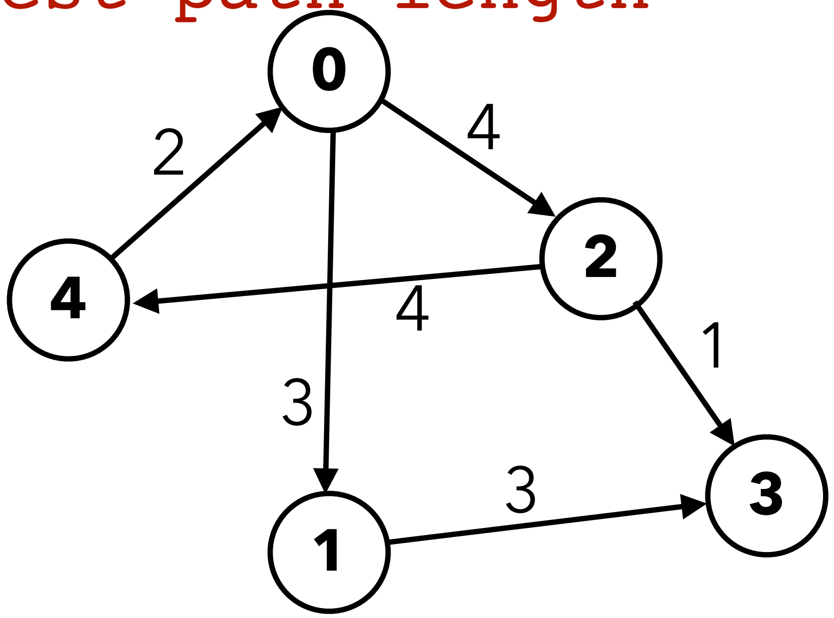
```

f: 0

Node	d
0	2
1	?
2	?
3	?
4	0

Settled set: {4, 0}

Frontier set: {}



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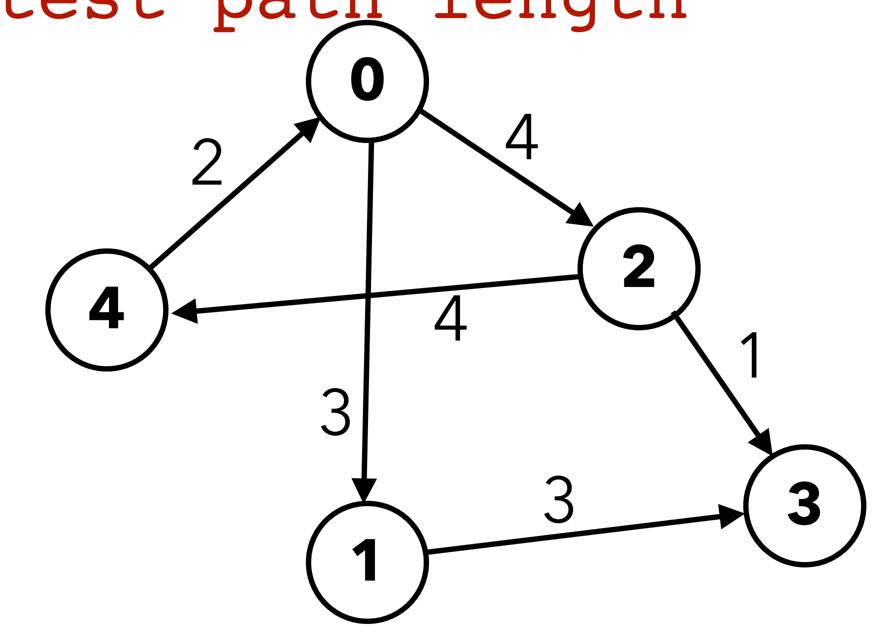
      add w to the frontier
    else if the path to w via f is shorter:
      update w's shortest path length
```

f: 0
w: 1

Node	d
0	2
1	**5**
2	?
3	?
4	0

Settled set: {4, 0}

Frontier set: {1}



`shortest-paths(4)`

Dijkstra's Shortest Paths: Execution

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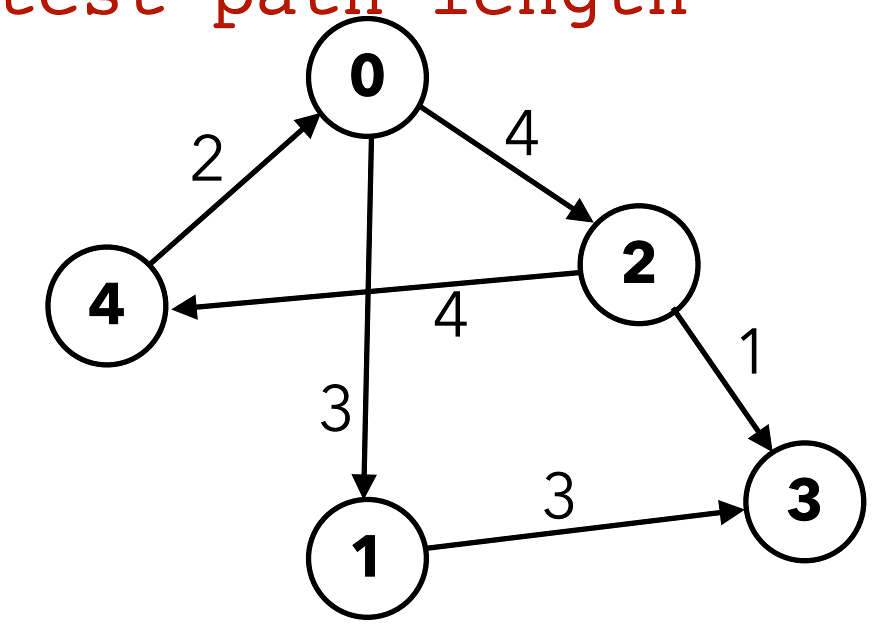
Initialize Settled to empty
Initialize Frontier to the start node
While the frontier isn't empty:
   move the node f with smallest d from F to S
    For each neighbor w of f:                  f: 0
      if we've never seen w before:            w: 2
        set its path length to f.d + wt(f,w)
        add w to the frontier
      else if the path to w via f is shorter:
        update w's shortest path length
```

Node	d
0	2
1	5
2	**6**
3	?
4	0

Settled set: {4, 0}

Frontier set: {1, 2}



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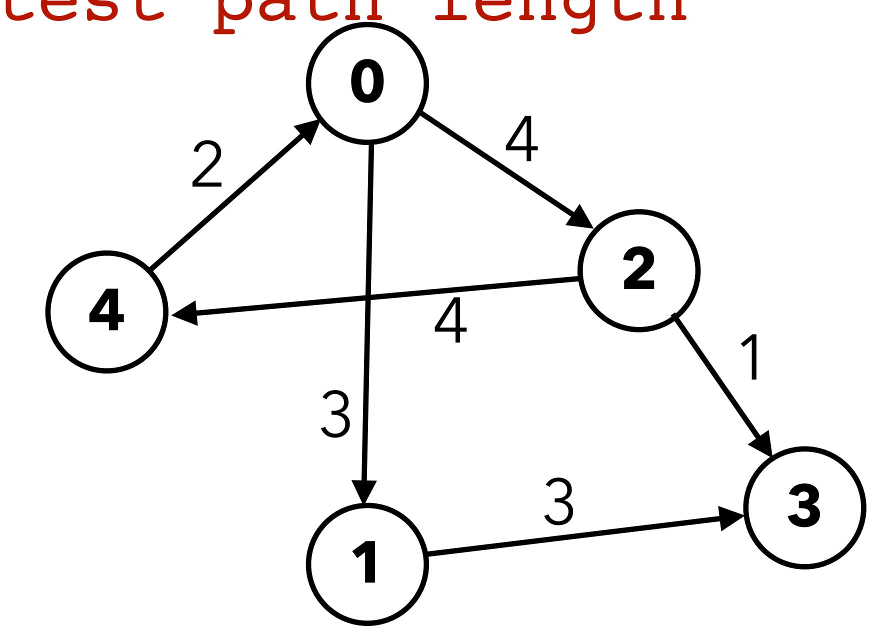
    else if the path to w via f is shorter:
      update w's shortest path length
```

f: 1

Node	d
0	2
1	5
2	6
3	**8**
4	0

Settled set: {4, 0, 1}

Frontier set: {2}



shortest-paths(4)

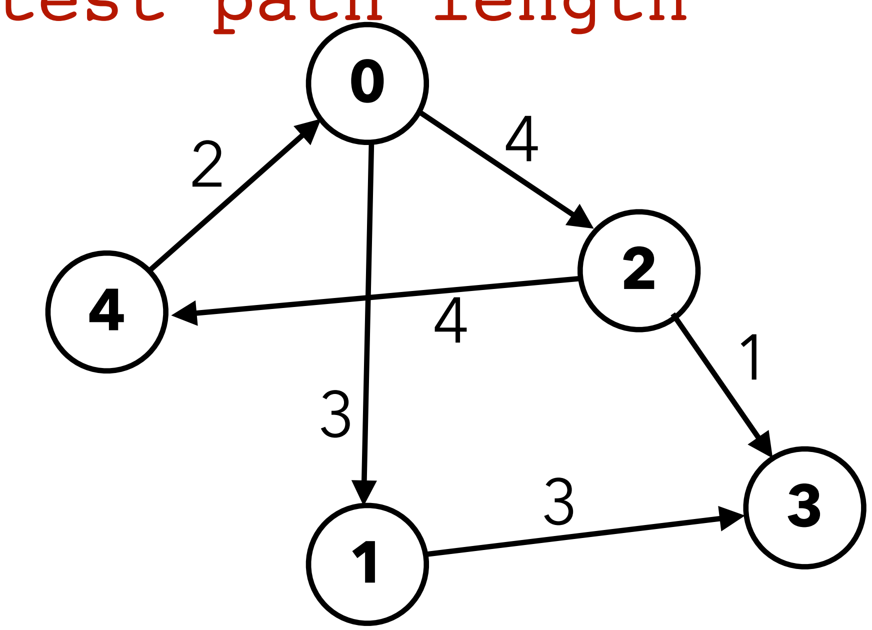
Dijkstra's Shortest Paths: Execution

Node	d
0	2
1	5
2	6
3	8
4	0

```
Initialize Settled to empty
Initialize Frontier to the start node
While the frontier isn't empty:
  move the node f with smallest d from F to S
  For each neighbor w of f:                    f: 1
    if we've never seen w before:              w: 3
      set its path length to f.d + wt(f,w)
      add w to the frontier
    else if the path to w via f is shorter:
      update w's shortest path length
```

Settled set: {4, 0, 1}

Frontier set: {2, 3}



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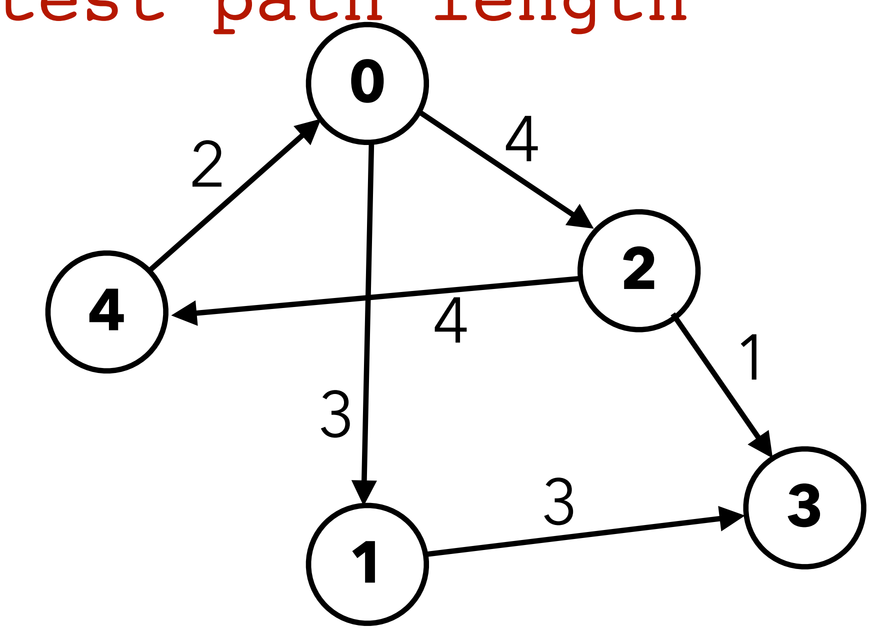
```

f: 2

Node	d
0	2
1	5
2	6
3	8
4	0

Settled set: {4, 0, 1, 2}

Frontier set: {3}



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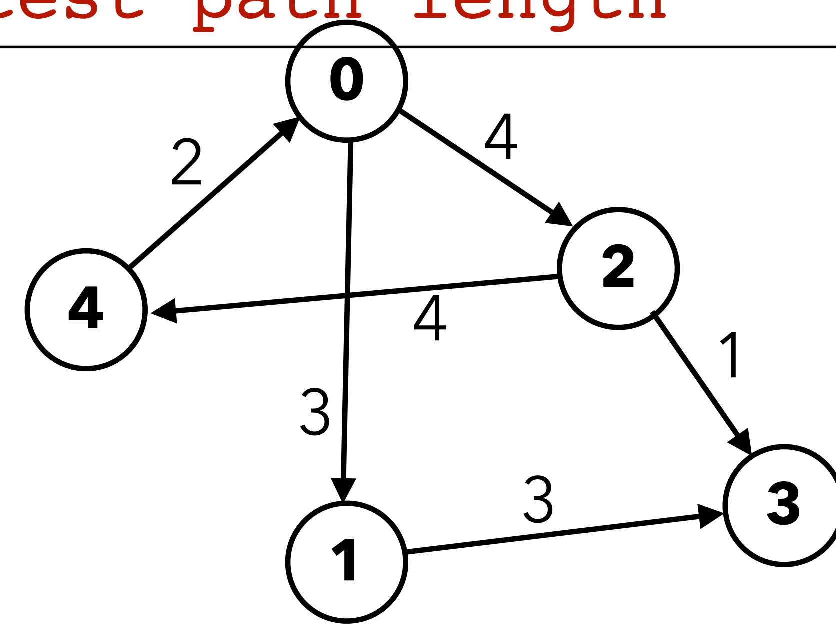
f: 2
w: 3

Node	d
0	2
1	5
2	6
3	**8**
4	0

Settled set: {4, 0, 1, 2}

Frontier set: {3}



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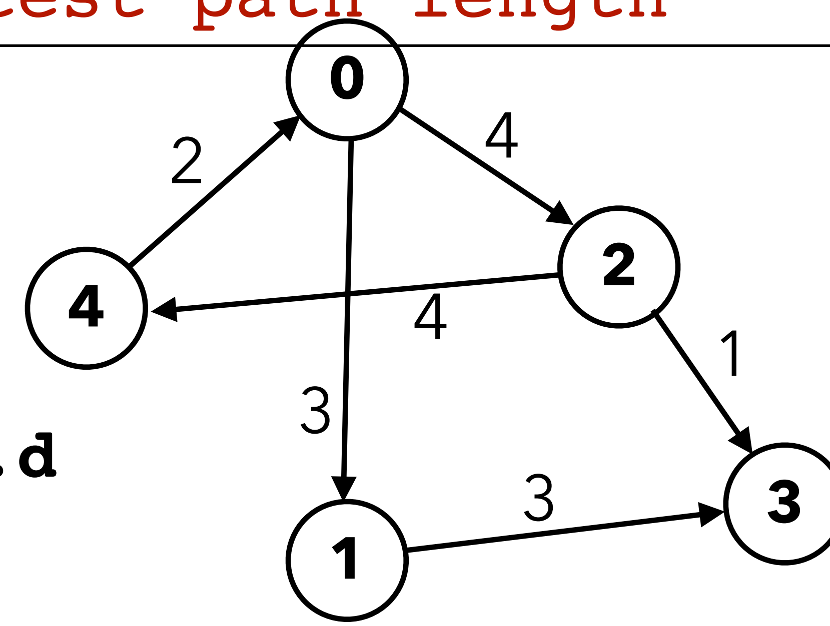
f: 2
w: 3

Node	d
0	2
1	5
2	6
3	**8**
4	0

Settled set: {4, 0, 1, 2}

Frontier set: {3}

$$2.d + wt(2,3) \overset{?}{<} 3.d$$



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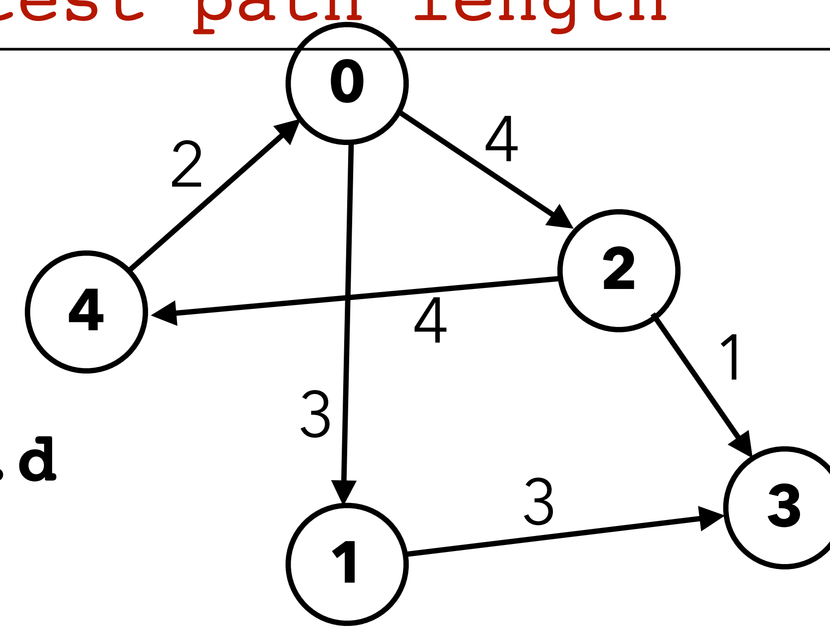
f: 2
w: 3

Node	d
0	2
1	5
2	6
3	**8**
4	0

Settled set: {4, 0, 1, 2}

Frontier set: {3}

$$2.d + wt(2,3) \overset{?}{<} 3.d$$
$$6 + 1 < 8$$



shortest-paths(4)

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```

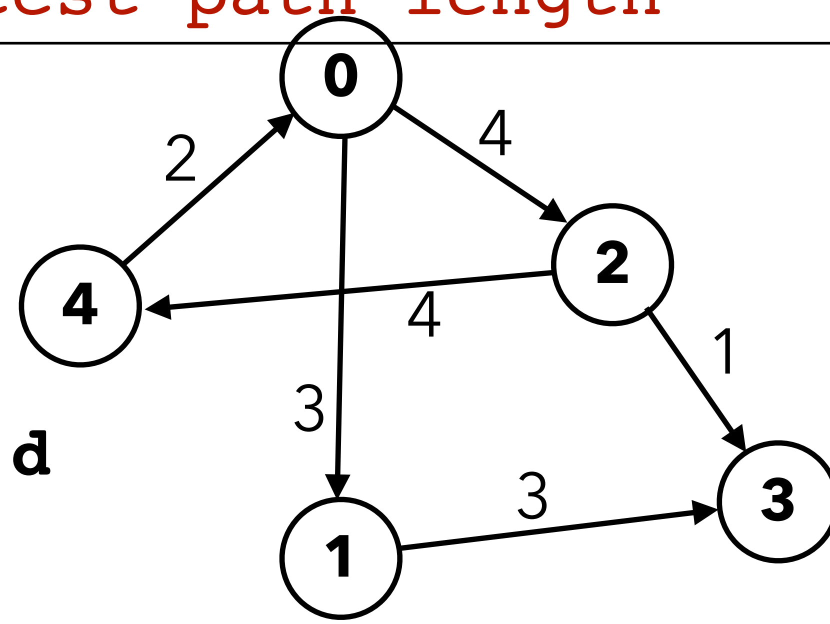
f: 2
w: 3

Node	d
0	2
1	5
2	6
3	**8**
4	0

Settled set: {4, 0, 1, 2}

Frontier set: {3}

$$2.d + wt(2,3) \overset{?}{<} 3.d$$
$$6 + 1 < 8$$
$$7 < 8$$



shortest-paths(4)

Dijkstra's Shortest Paths: Execution

```
Initialize Settled to empty
Initialize Frontier to the start node
While the frontier isn't empty:
  move the node f with smallest d from F to S
  For each neighbor w of f:
    if we've never seen w before:
      set its path length to f.d + wt(f,w)
      add w to the frontier
    else if the path to w via f is shorter:
      update w's shortest path length
```

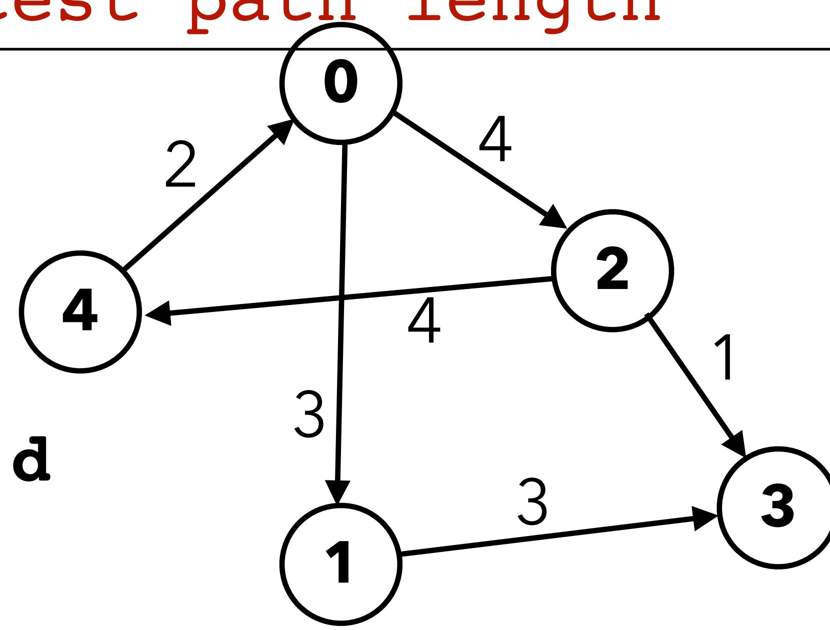
f: 2
w: 3

Node	d
0	2
1	5
2	6
3	**7**
4	0

Settled set: {4, 0, 1, 2}

Frontier set: {3}

```
                ?
2.d + wt(2,3)   <  3.d
 6  +   1       <  8
            7   <  8
```



shortest-paths(4)

Dijkstra's Shortest Paths: Execution

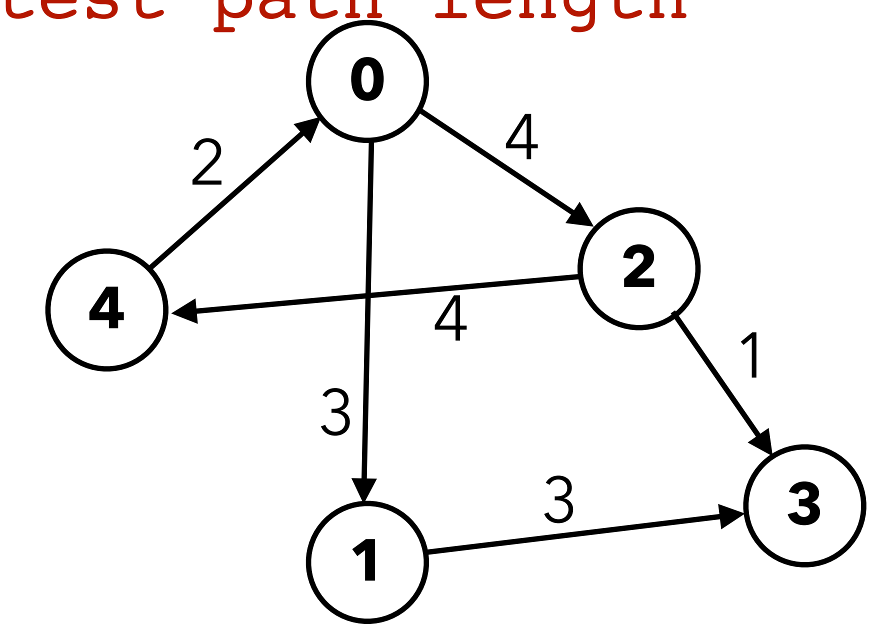
```
Initialize Settled to empty
Initialize Frontier to the start node
While the frontier isn't empty:
   move the node f with smallest d from F to S
   For each neighbor w of f:
      if we've never seen w before:
         set its path length to f.d + wt(f,w)
         add w to the frontier
      else if the path to w via f is shorter:
         update w's shortest path length
```

f: 3

Node	d
0	2
1	5
2	6
3	7
4	0

Settled set: {4, 0, 1, 2, 3}

Frontier set: {} Empty => done!



shortest-paths(4)