CSCI 241

Scott Wehrwein

Runtime Analysis:
Case study - Binary Search
Best-, Worst-, and Average-case Analysis

Goals

Understand the runtime analysis of binary search.

Know how to perform best-case, worst-case, and average-case runtime analysis.

Runtime of Binary Search

Let N = A.length and assume x is not in A.

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Strategy:

1. Identify constant-time operations.

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so, O(L) ...but what is L?

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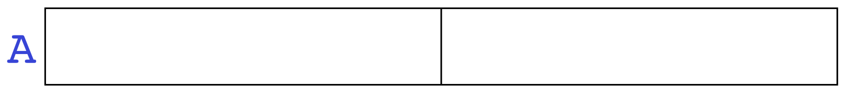

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A	can be here	can't be here

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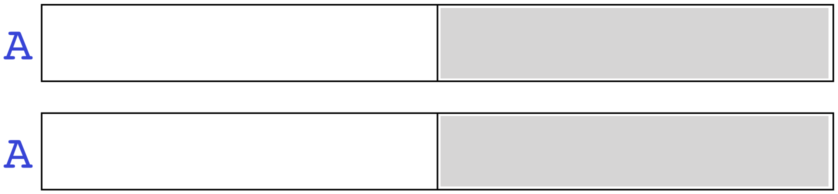

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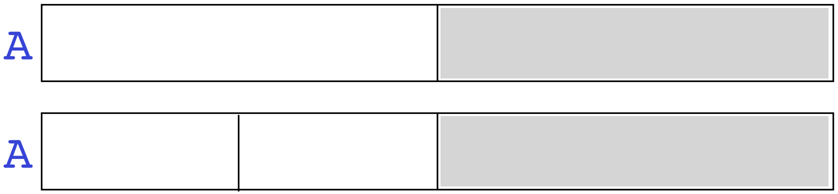

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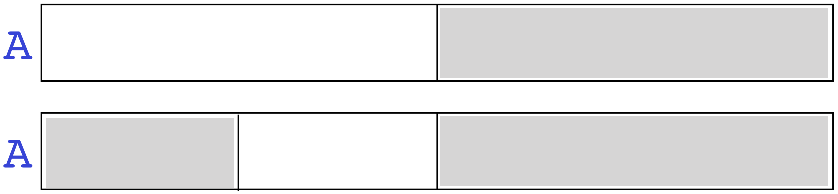

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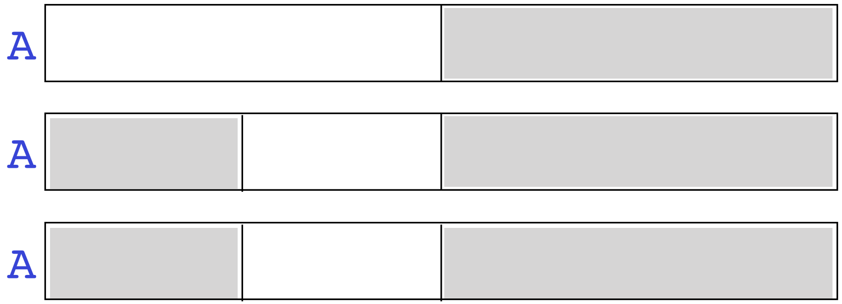

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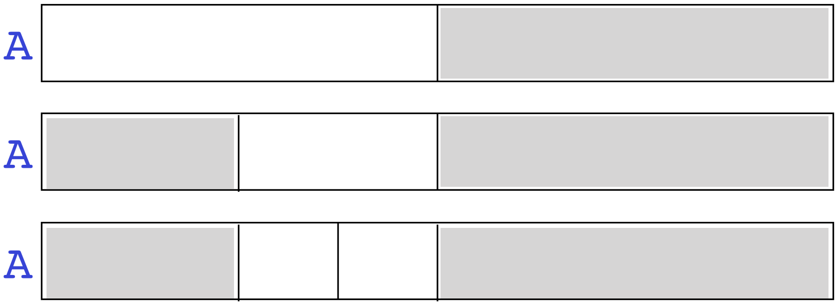

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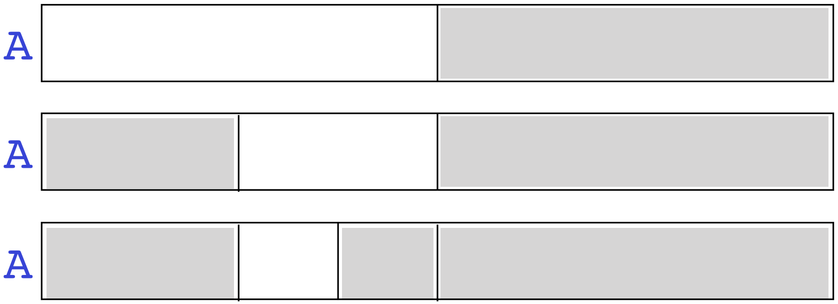

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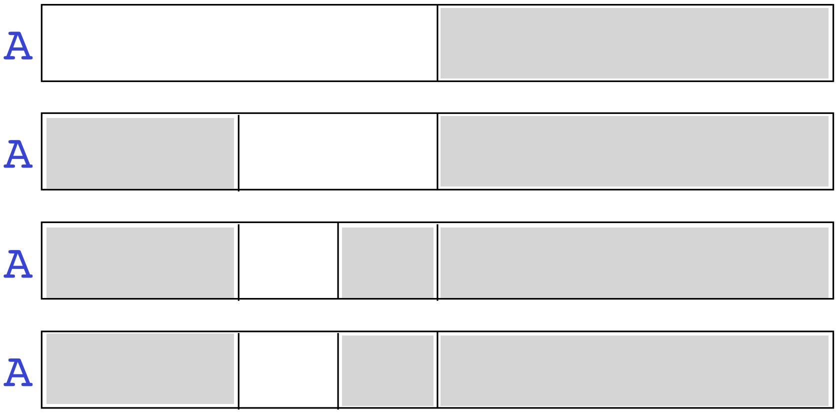

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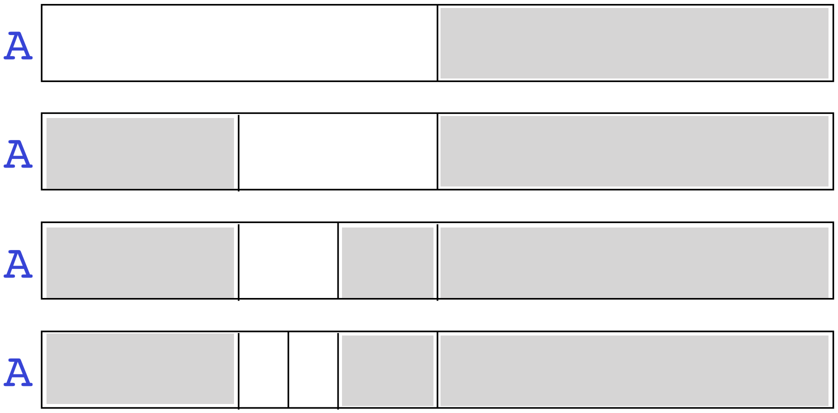

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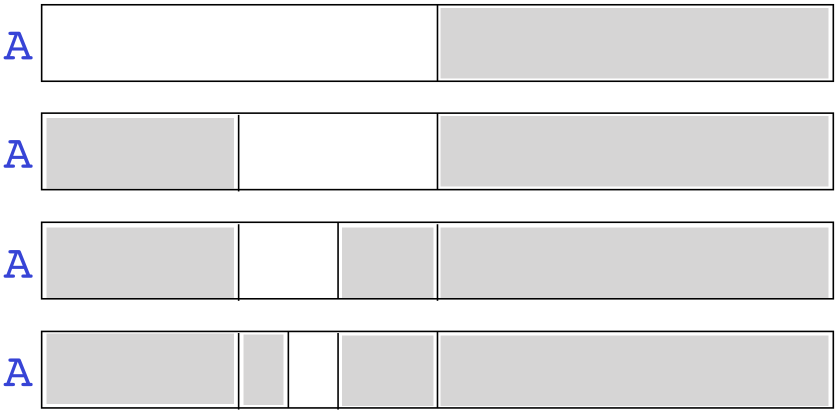

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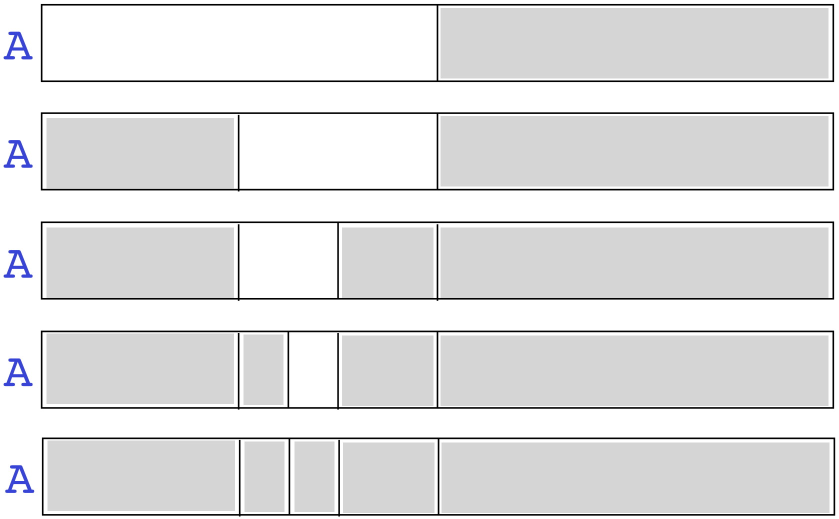
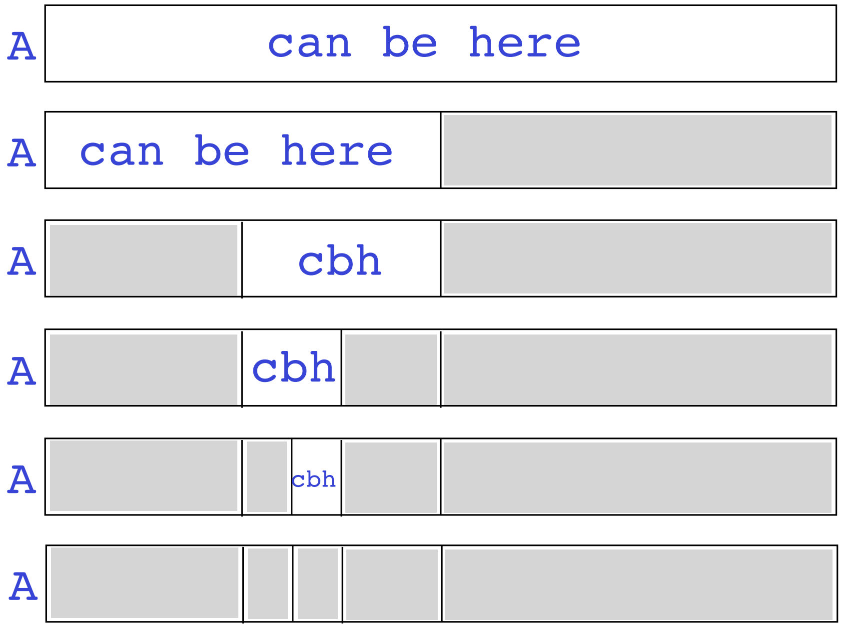


Size of the "can be here" region:



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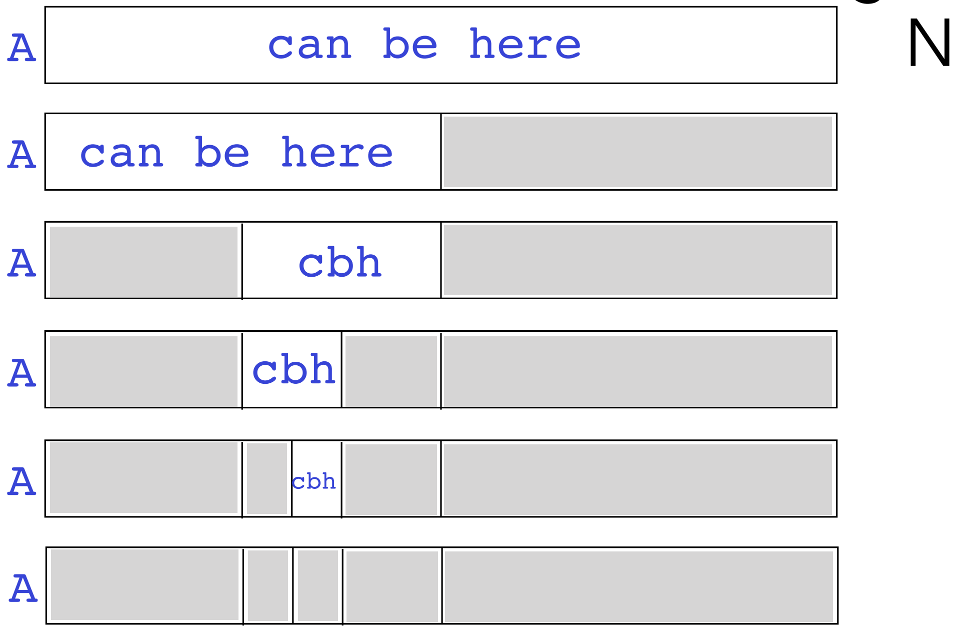

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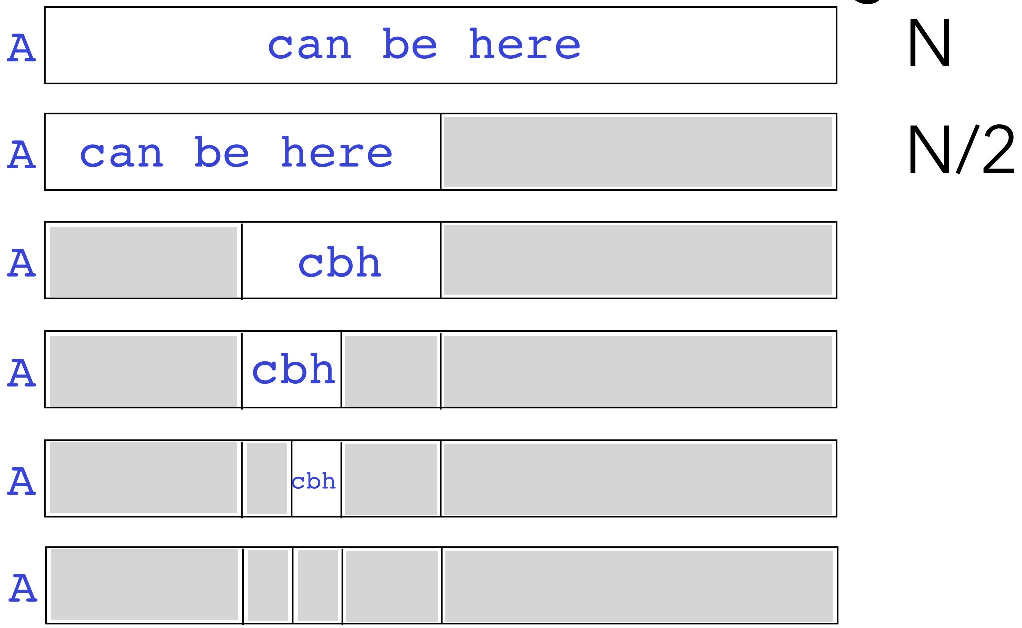

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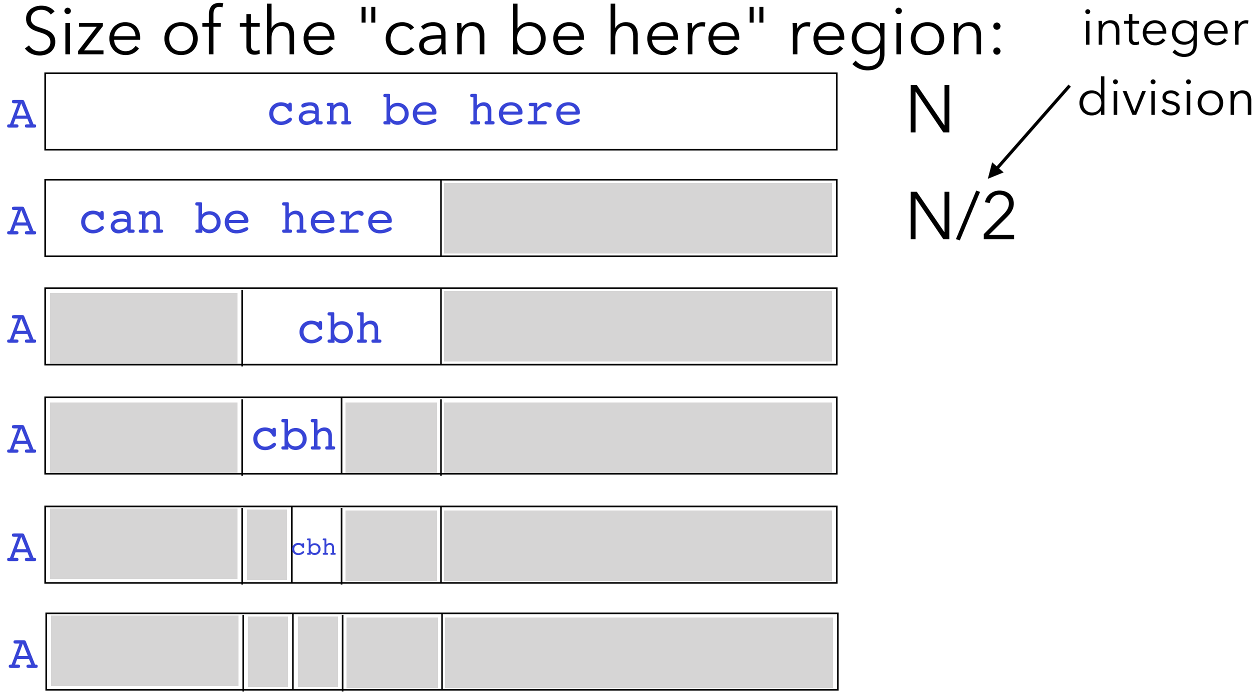

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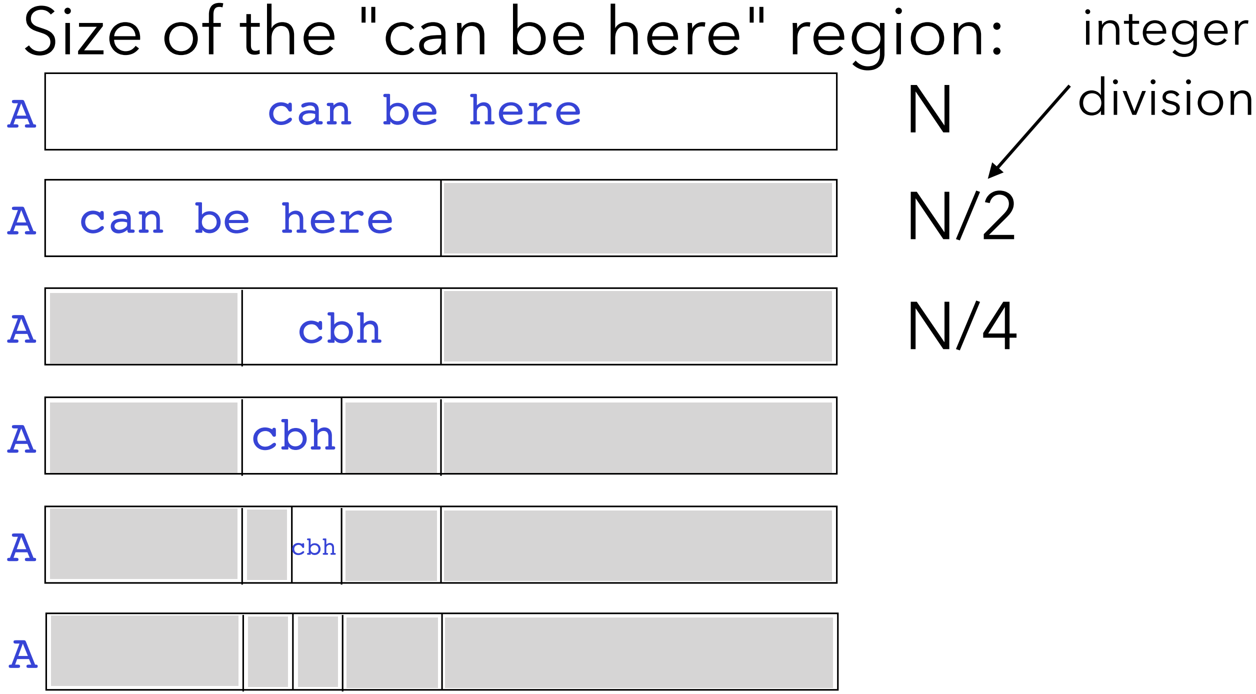

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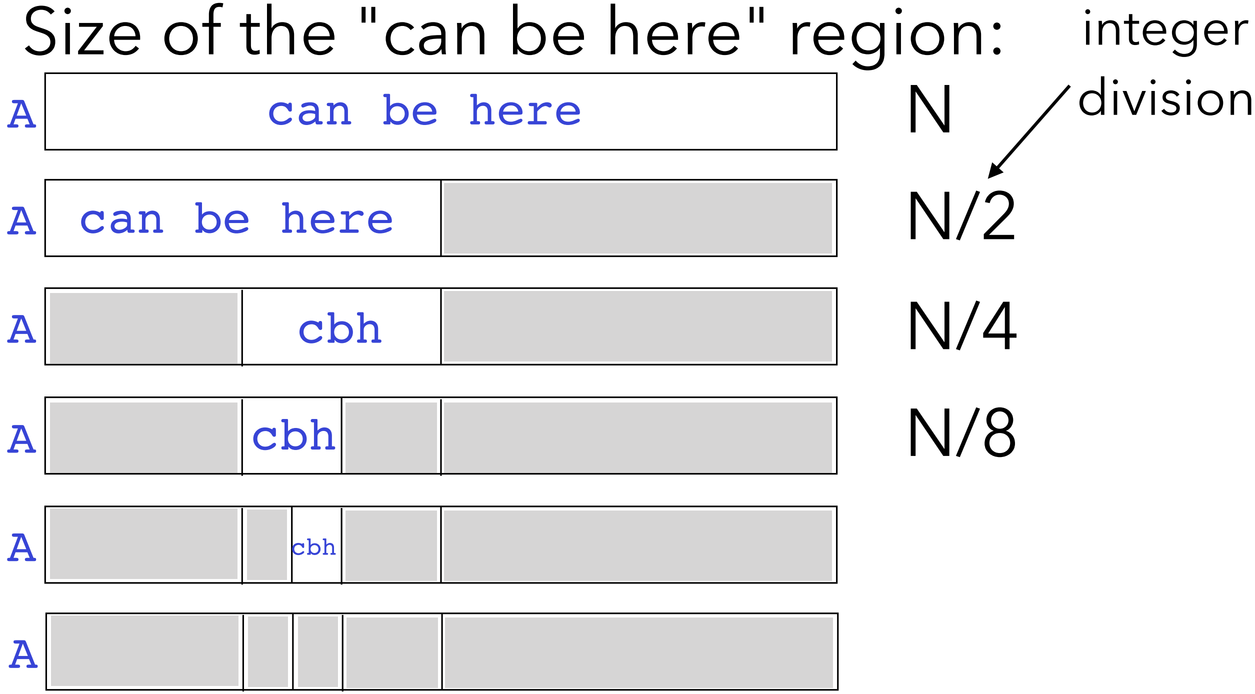

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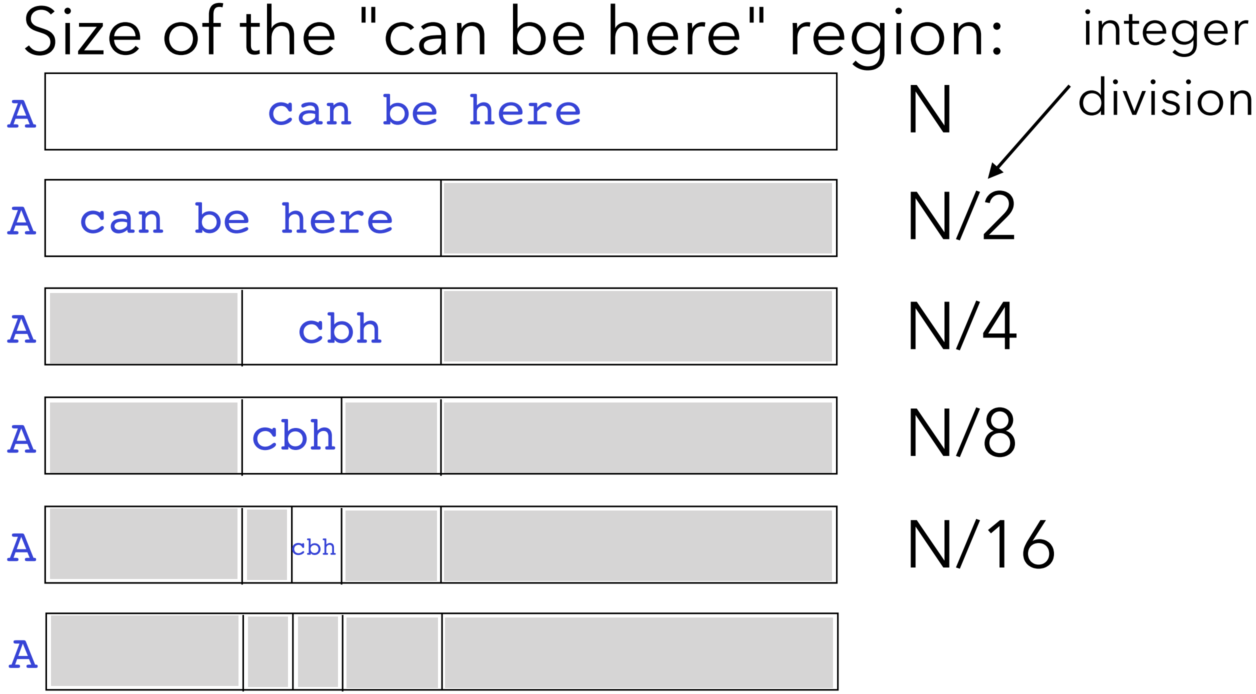

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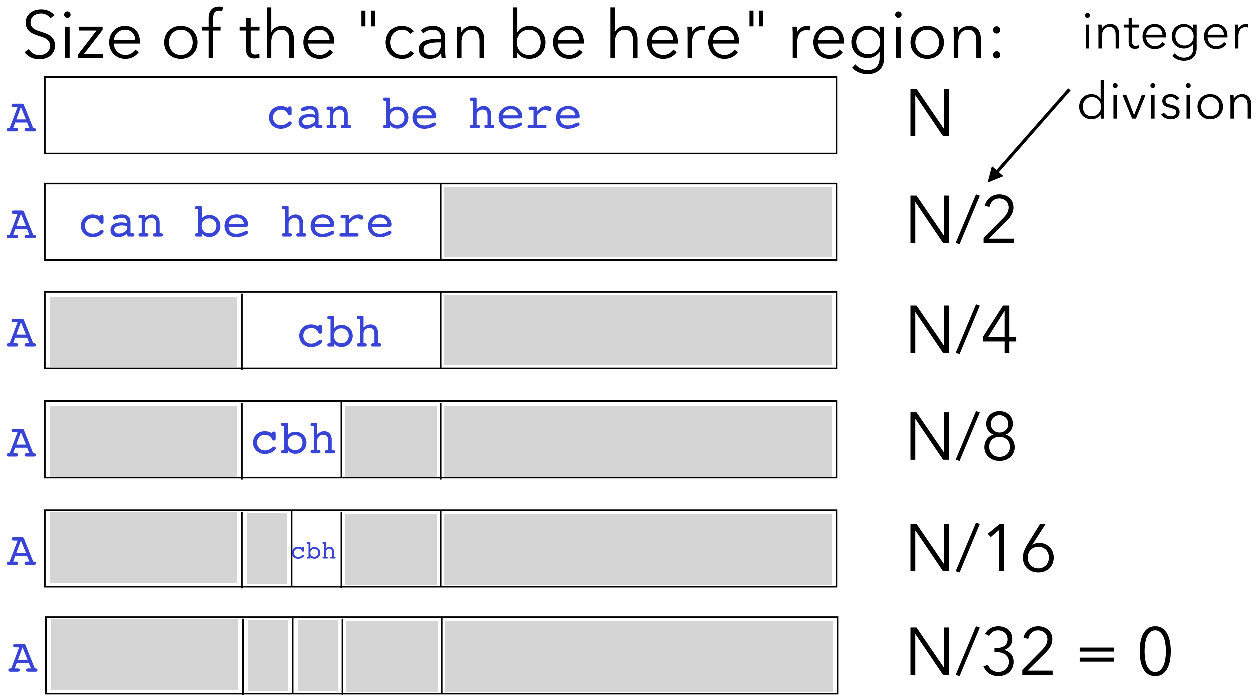

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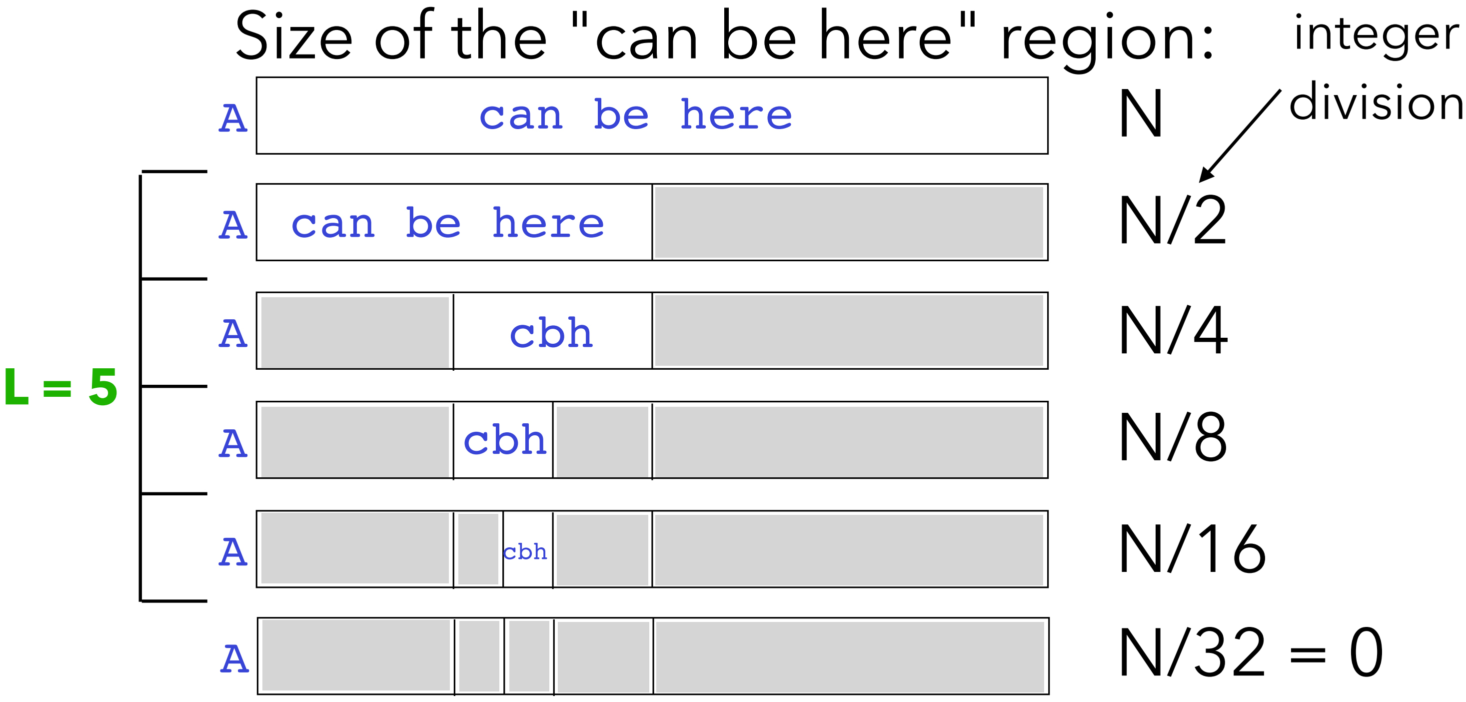

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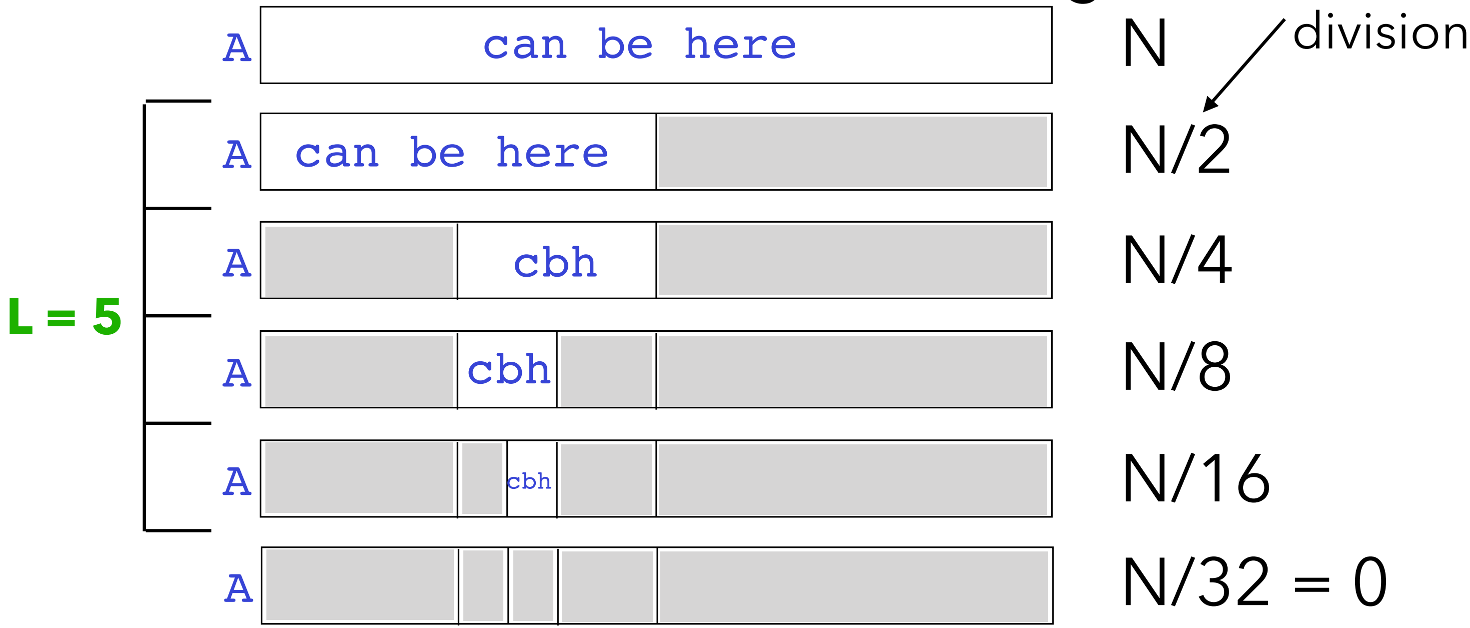

So, L is the answer to:

How many times can I divide N by 2 before it becomes 0?

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Or equivalently, using real (non-integer) division:

How many times can I divide N by 2 before it's less than 1?

$$\frac{N}{2^L} < 1$$

$$N < 2^L$$

$$\log_2 N < \log_2 2^L$$

$$\log_2 N < L$$

$$\lceil \log_2 N \rceil = L$$

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$5 \log_2(n) + 4$ is O(log n)

Aside: Bases of Logarithms

Fact: converting from one base to another only requires multiplication by a constant.

$$\log_b(x) = \frac{\log_d(x)}{\log_d(b)}$$

$$\log_b(x) = \cancel{\frac{1}{\log_d(b)}} \cdot \log_d(x)$$

$$\cancel{c}$$

Aside: Bases of Logarithms

Corollary: the base of the logarithm doesn't affect the big-O class.

$$c \cdot \log_2(n) = \log_{10} n$$

$O(\log_2 n)$ is same as $O(\log_{10} n)$

Aside: Bases of Logarithms

Convention: We can use logs without specifying a base in big-O notation.

$$O(\log_2 n)$$

$$\downarrow$$

$$O(\log n)$$

Which algorithm is better?

Suppose you have two different algorithms that solve the same problem. For example, *search a sorted array*.

```
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The operation count depends on the data!

- If `x` is at `A[0]`, runtime is O(1)
- If `x` is not in `A`, runtime is O(N)
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  (one possible notion of average-case runtime)