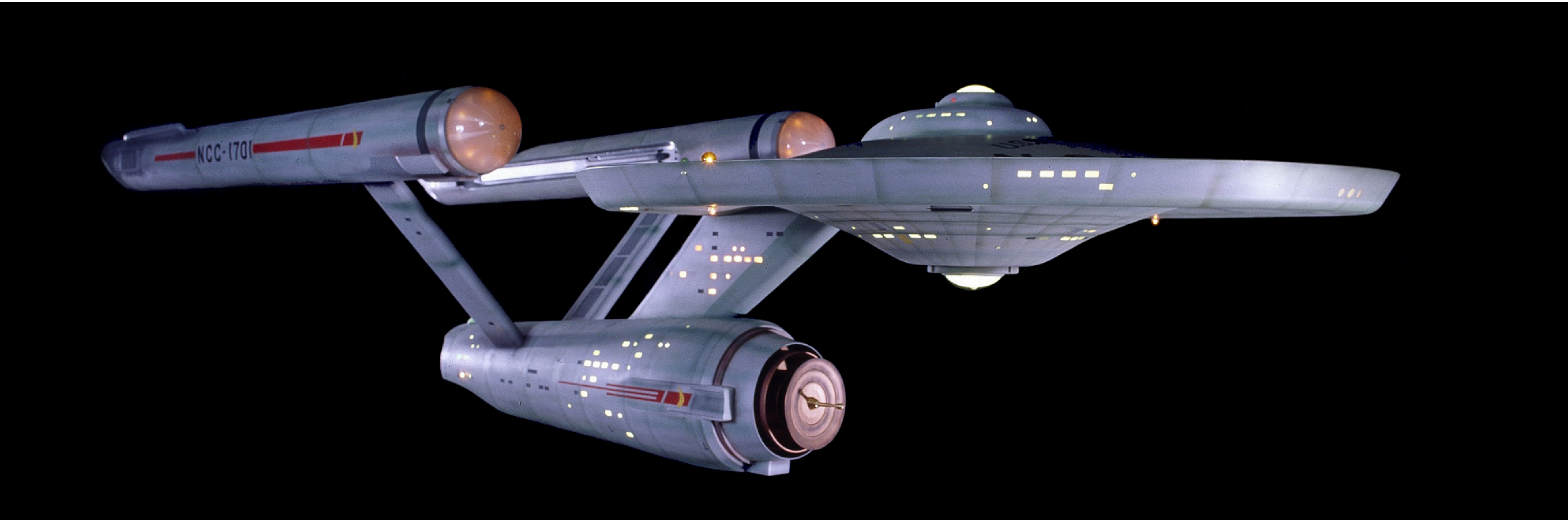


CSCI 241

Lecture 20

Dijkstra's Algorithm:

Implementation, Proof of Correctness

Happenings

Today: AWC-organized panel discussion with recent female alumnae working in the tech industry. AW 210 at 4:00.

Tuesday, 3/12 – AWC Bake Sale – 10 am to 2 pm in the CF 1st Floor Lobby

Wednesday, 3/13 – Peer Lecture Series: Machine Learning – 5 pm in CF 420

Thursday, 3/14 – CS Study Break: Bingo and Brownies! – 3 pm in the CF 4th Floor Foyer

COMPUTER SCIENCE

B I N G O

STUDY BREAK

Join us for brownies and bingo!

3 - 4 PM

THURSDAY, MARCH 14

CF 4TH FLOOR FOYER

Bingo called by Aran Clauson

For disability resources, contact 360-650-3083

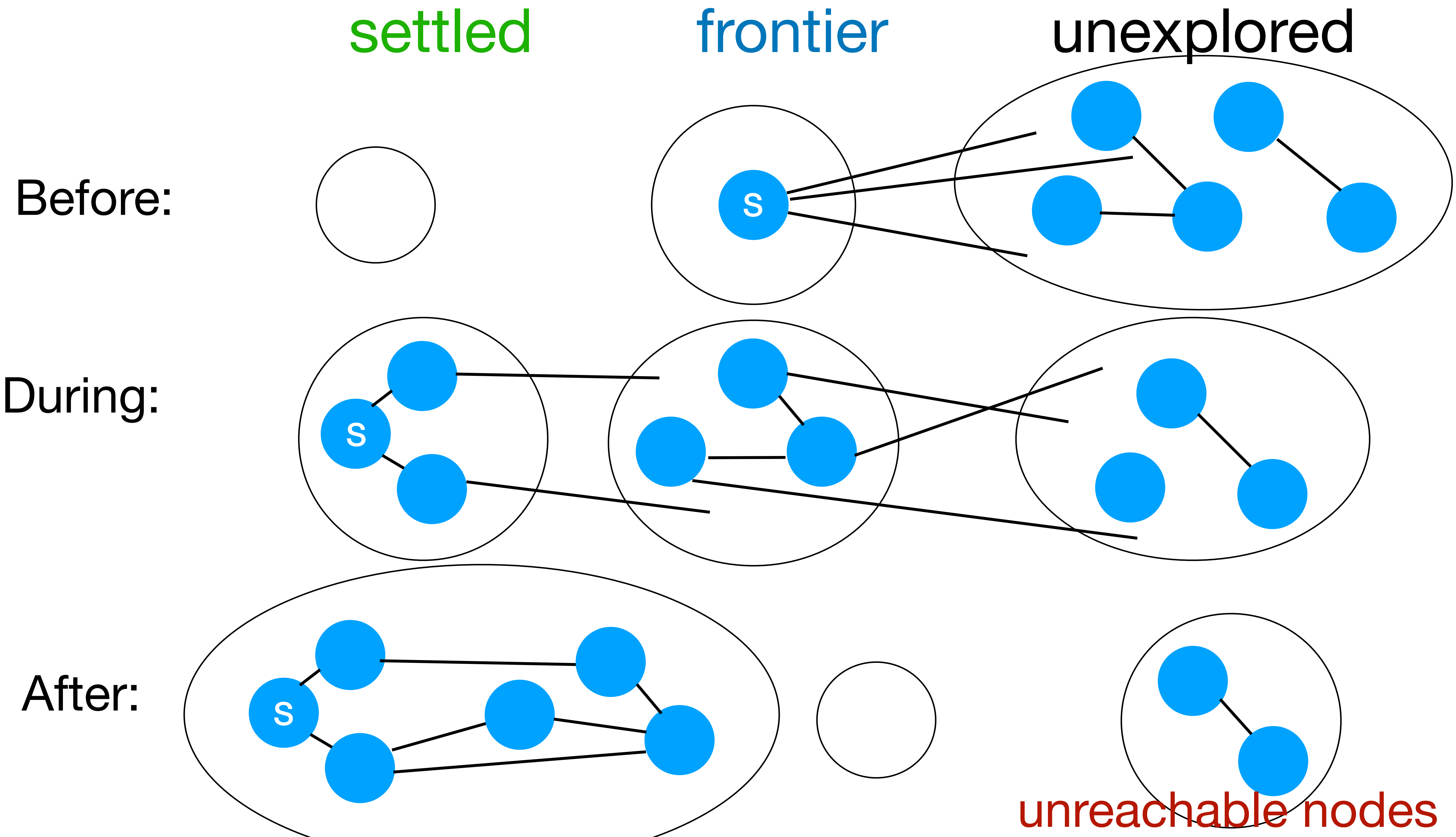
Announcements

- No quiz today.
- Quiz Monday instead.

Goals

- Be able to run Dijkstra's algorithm on paper.
- Know how to implement Dijkstra efficiently (as in A4)
- Know how to augment the algorithm to keep backpointers in order to reconstruct the sequence of nodes in a shortest path.

Dijkstra’s Shortest Paths: Cartoon



Dijkstra’s Shortest Paths: Intuition

- Intuition: explore nodes kinda like BFS.
- There are three kinds of nodes:
  - Settled - nodes for which we know the actual shortest path.
  - Frontier - nodes that have been visited but we don’t necessarily have their actual shortest path
  - Unexplored - all other nodes.
- Each node n keeps track of `n.d`, the length of the shortest known known path from start.
- We may discover a shorter path to a frontier node than the one we’ve found already - if so, update `n.d`.

Dijkstra's Shortest Paths: High-Level Algorithm

```
Initialize Settled to empty
Initialize Frontier to the start node
While the frontier isn't empty:
  move the node f with smallest d from F to S
  For each neighbor w of f:
    if we've never seen w before:
      set its path length
      add it to frontier
  else if the path to w via f is shorter:
      update w's shortest path length
```

Dijkstra's Shortest Paths: High-Level Algorithm

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While the frontier isn't empty:
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         add it to frontier
   else if the path to w via f is shorter:
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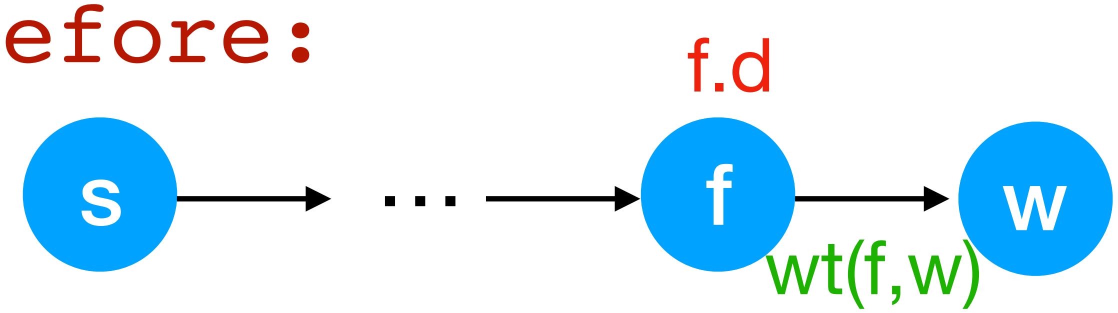

Dijkstra’s Shortest Paths: High-Level Algorithm

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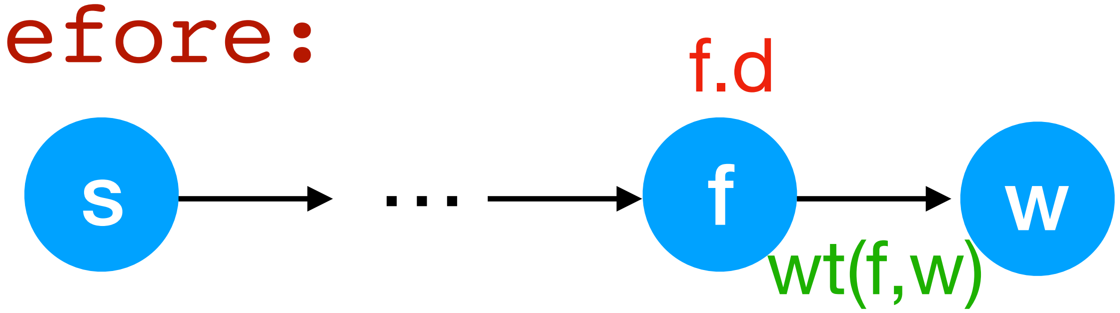

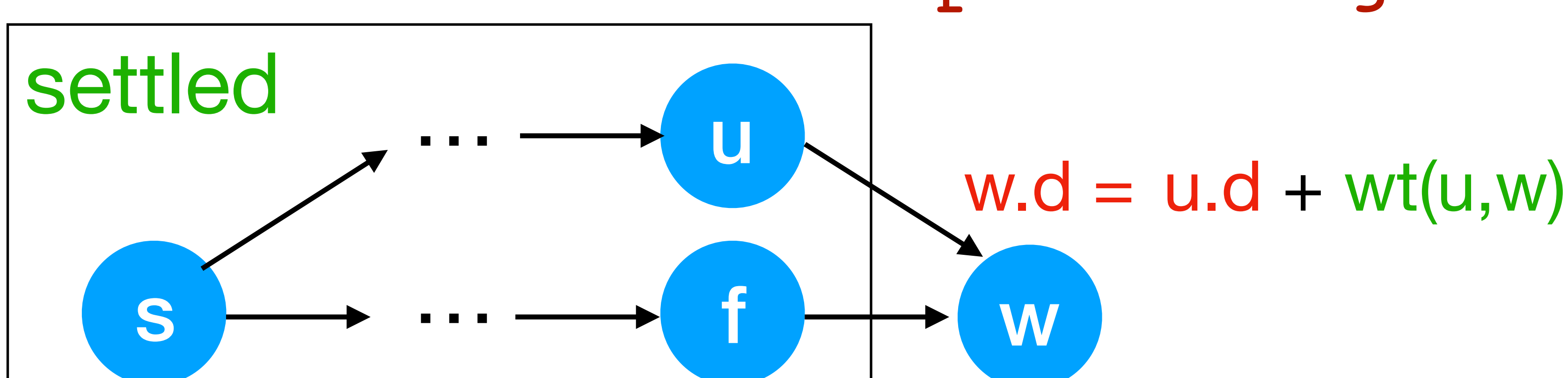

Dijkstra’s Shortest Paths: High-Level Algorithm

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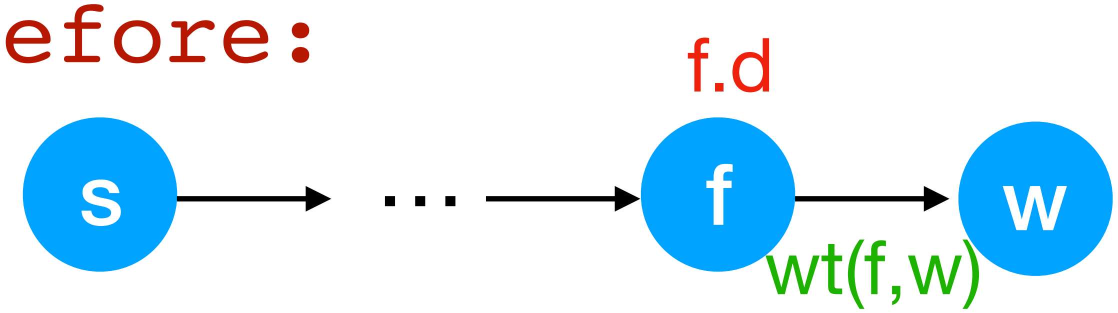
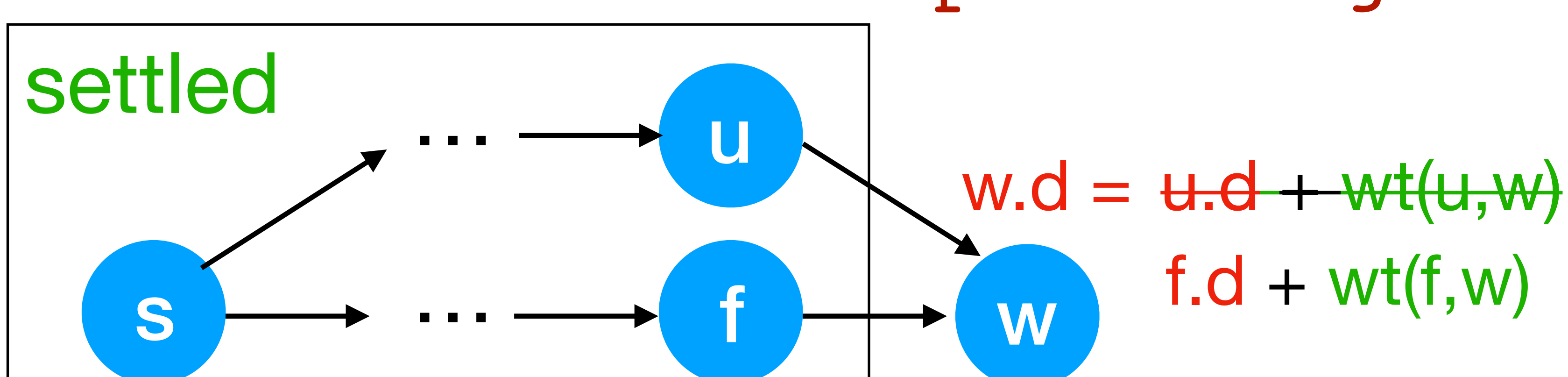




Dijkstra's Shortest Paths: Execution

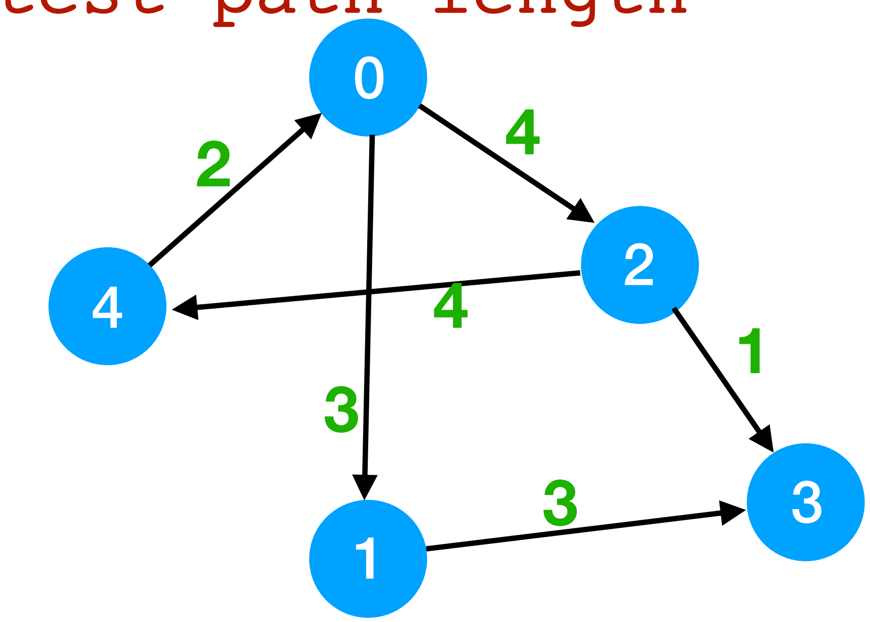
Best known distances:

Node	d
0	?
1	?
2	?
3	?
4	?

```
Initialize Settled to empty
Initialize Frontier to the start node
While the frontier isn't empty:
  move the node f with smallest d from F to S
  For each neighbor w of f:
    if we've never seen w before:
      set its path length to f.d + wt(f,w)
      add w to the frontier
    else if the path to w via f is shorter:
      update w's shortest path length
```

Settled set:

Frontier set:



shortest-paths(4)

Dijkstra's Shortest Paths: Execution

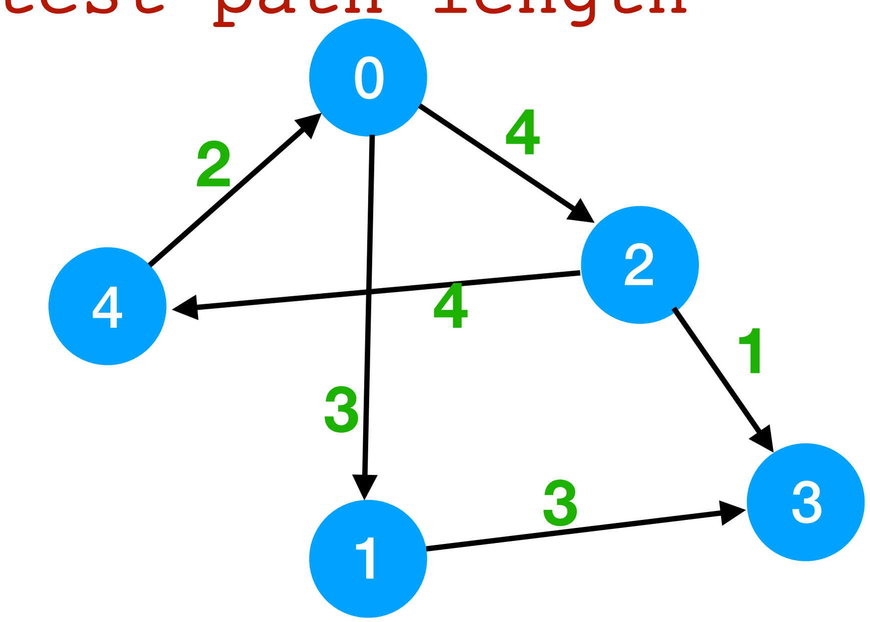
Best known distances:

Node	d
0	?
1	?
2	?
3	?
4	**0**

```
Initialize Settled to empty
Initialize Frontier to the start node
While the frontier isn't empty:
  move the node f with smallest d from F to S
  For each neighbor w of f:
    if we've never seen w before:
      set its path length to f.d + wt(f,w)
      add w to the frontier
    else if the path to w via f is shorter:
      update w's shortest path length
```

Settled set: {}

Frontier set: {4}



shortest-paths(4)

Dijkstra's Shortest Paths: Execution

Best known distances:

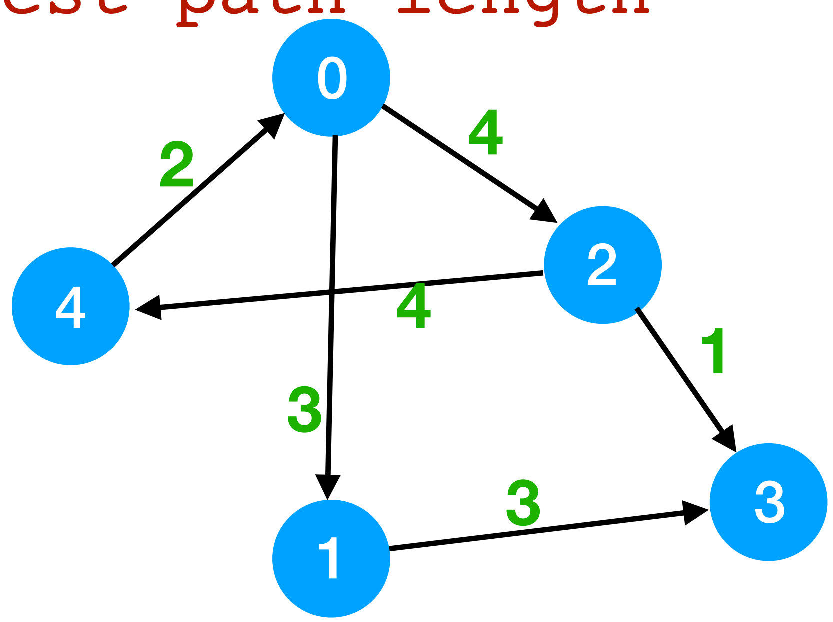
Node	d
0	?
1	?
2	?
3	?
4	0

```
Initialize Settled to empty
Initialize Frontier to the start node
While the frontier isn't empty:
  move the node f with smallest d from F to S
  For each neighbor w of f:
    if we've never seen w before:
      set its path length to f.d + wt(f,w)
      add w to the frontier
    else if the path to w via f is shorter:
      update w's shortest path length
```

f: 4

Settled set: {4}

Frontier set: {}



shortest-paths(4)

Dijkstra’s Shortest Paths: Execution

Best known distances:

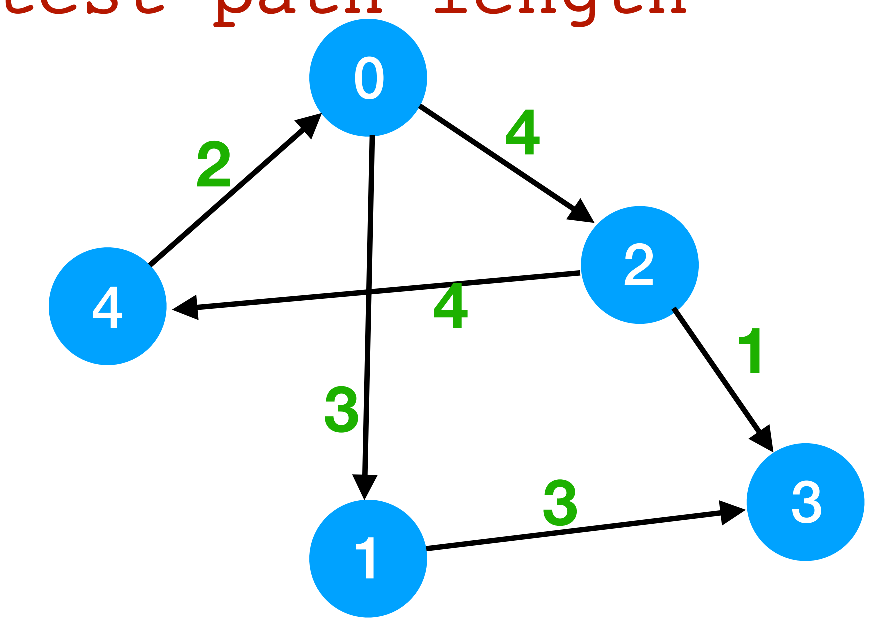
Node	d
0	2
1	?
2	?
3	?
4	0

```
Initialize Settled to empty
Initialize Frontier to the start node
While the frontier isn't empty:
   move the node f with smallest d from F to S
   For each neighbor w of f:
     if we've never seen w before:
       set its path length to f.d + wt(f,w)
       add w to the frontier
     else if the path to w via f is shorter:
       update w's shortest path length
```

f: 4
w: 0

Settled set: {4}

Frontier set: {0}



shortest-paths(4)

Dijkstra's Shortest Paths: Execution

Best known distances:

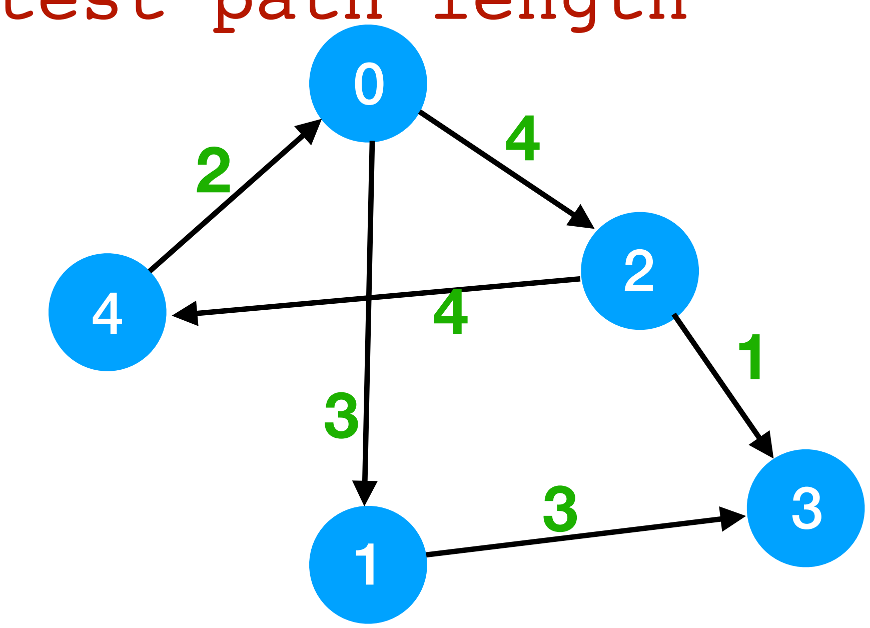
Node	d
0	2
1	?
2	?
3	?
4	0

```
Initialize Settled to empty
Initialize Frontier to the start node
While the frontier isn't empty:
  move the node f with smallest d from F to S
  For each neighbor w of f:
    if we've never seen w before:
      set its path length to f.d + wt(f,w)
      add w to the frontier
    else if the path to w via f is shorter:
      update w's shortest path length
```

f: 0

Settled set: {4, 0}

Frontier set: {}



shortest-paths(4)

Dijkstra's Shortest Paths: Execution

Best known distances:

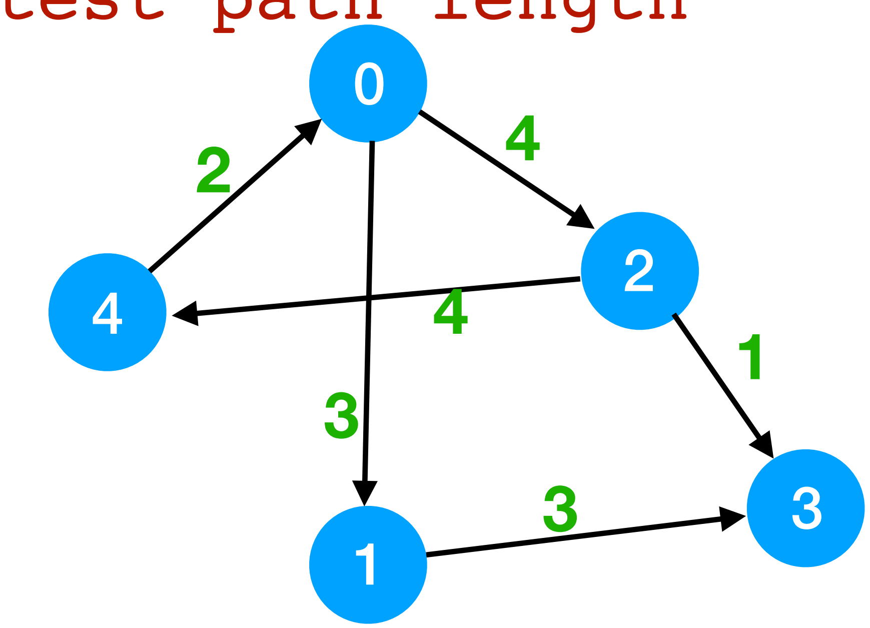
Node	d
0	2
1	**5**
2	?
3	?
4	0

```
Initialize Settled to empty
Initialize Frontier to the start node
While the frontier isn't empty:
  move the node f with smallest d from F to S
  For each neighbor w of f:
    if we've never seen w before:
      set its path length to f.d + wt(f,w)
      add w to the frontier
    else if the path to w via f is shorter:
      update w's shortest path length
```

f: 0
w: 1

Settled set: {4, 0}

Frontier set: {1}



shortest-paths(4)

Dijkstra's Shortest Paths: Execution

Best known distances:

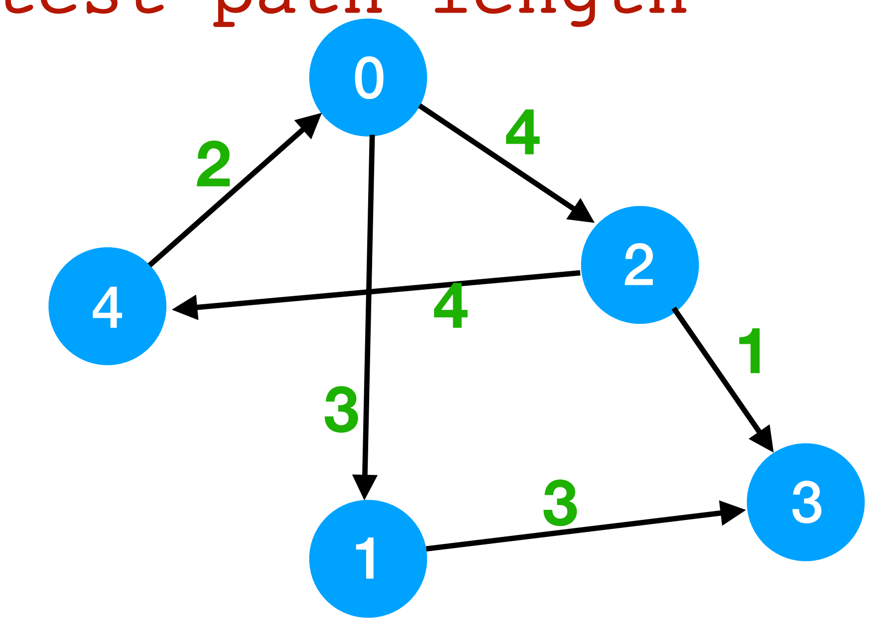
Node	d
0	2
1	5
2	**6**
3	?
4	0

```
Initialize Settled to empty
Initialize Frontier to the start node
While the frontier isn't empty:
  move the node f with smallest d from F to S
  For each neighbor w of f:
    if we've never seen w before:
      set its path length to f.d + wt(f,w)
      add w to the frontier
    else if the path to w via f is shorter:
      update w's shortest path length
```

f: 0
w: 2

Settled set: {4, 0}

Frontier set: {1, 2}



shortest-paths(4)

Dijkstra's Shortest Paths: Execution

Best known distances:

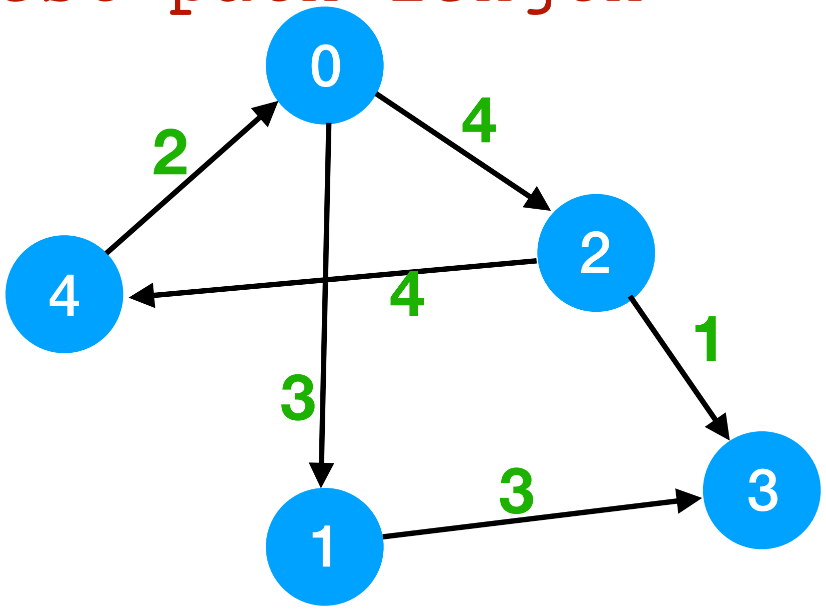
Node	d
0	2
1	5
2	6
3	**8**
4	0

```
Initialize Settled to empty
Initialize Frontier to the start node
While the frontier isn't empty:
   move the node f with smallest d from F to S
   For each neighbor w of f:
      if we've never seen w before:
         set its path length to f.d + wt(f,w)
         add w to the frontier
      else if the path to w via f is shorter:
         update w's shortest path length
```

f: 1

Settled set: {4, 0, 1}

Frontier set: {2}



shortest-paths(4)

Dijkstra's Shortest Paths: Execution

Best known distances:

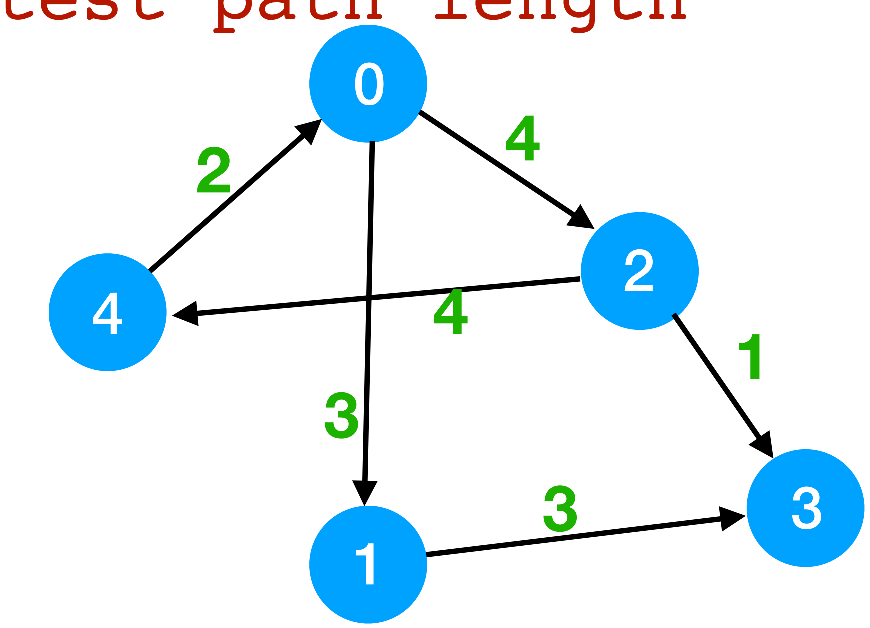
Node	d
0	2
1	5
2	6
3	8
4	0

```
Initialize Settled to empty
Initialize Frontier to the start node
While the frontier isn't empty:
   move the node f with smallest d from F to S
    For each neighbor w of f:
      if we've never seen w before:
        set its path length to f.d + wt(f,w)
        add w to the frontier
      else if the path to w via f is shorter:
        update w's shortest path length
```

f: 1
w: 3

Settled set: {4, 0, 1}

Frontier set: {2, 3}



shortest-paths(4)

Dijkstra’s Shortest Paths: Execution

Best known distances:

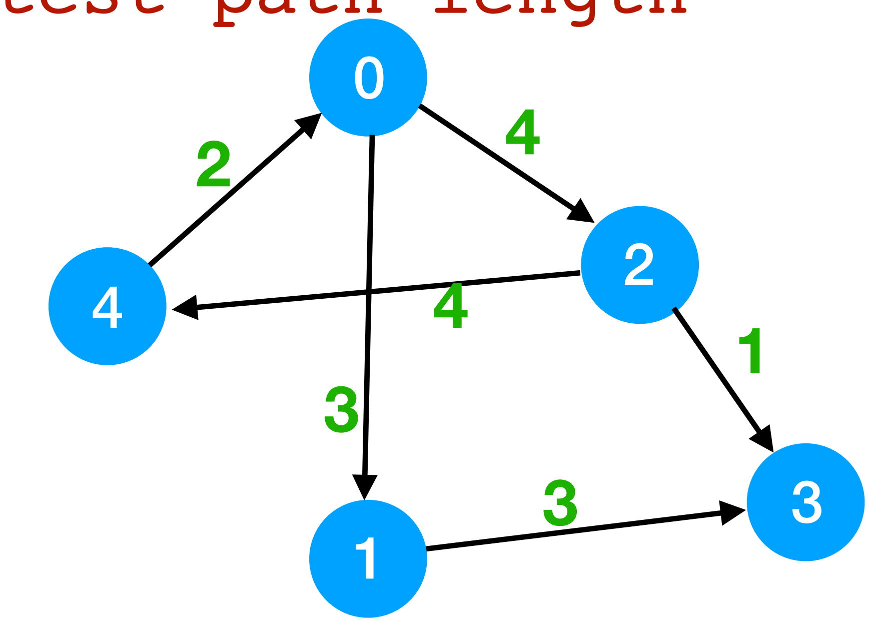
Node	d
0	2
1	5
2	6
3	8
4	0

```
Initialize Settled to empty
Initialize Frontier to the start node
While the frontier isn’t empty:
  move the node f with smallest d from F to S
  For each neighbor w of f:
    if we’ve never seen w before:
      set its path length to f.d + wt(f,w)
      add w to the frontier
    else if the path to w via f is shorter:
      update w’s shortest path length
```

f: 2

Settled set: {4, 0, 1, 2}

Frontier set: {3}



shortest-paths(4)

Dijkstra's Shortest Paths: Execution

Best known distances:

Node	d
0	2
1	5
2	6
3	7
4	0

```
Initialize Settled to empty
Initialize Frontier to the start node
While the frontier isn't empty:
  move the node f with smallest d from F to S
  For each neighbor w of f:
    if we've never seen w before:
      set its path length to f.d + wt(f,w)
      add w to the frontier
    else if the path to w via f is shorter:
      update w's shortest path length
```

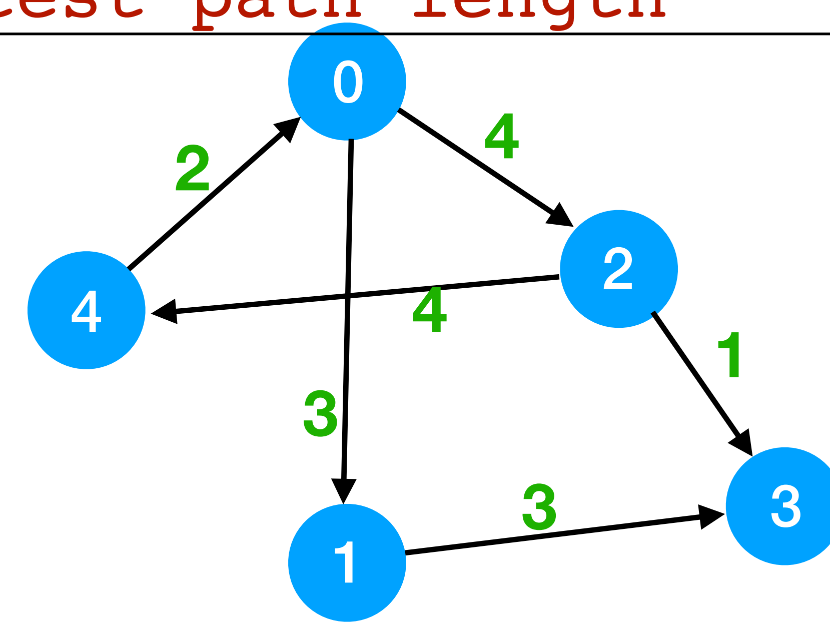
f: 2
w: 3

2.d + wt(2,3) < 3.d

7 < 8

Settled set: {4, 0, 1, 2}

Frontier set: {3}



shortest-paths(4)

Dijkstra's Shortest Paths: Execution

Best known distances:

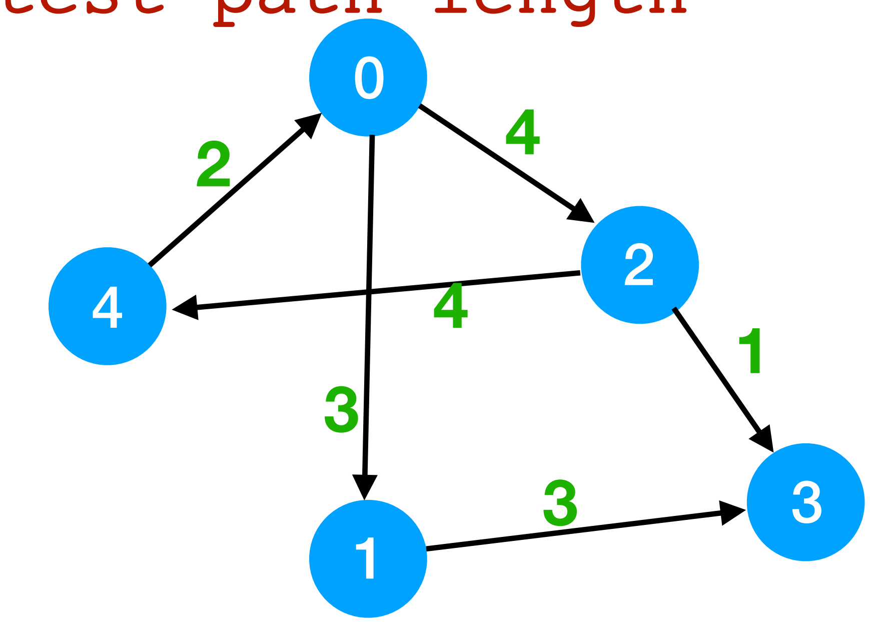
Node	d
0	2
1	5
2	6
3	7
4	0

```
Initialize Settled to empty
Initialize Frontier to the start node
While the frontier isn't empty:
  move the node f with smallest d from F to S
  For each neighbor w of f:
    if we've never seen w before:
      set its path length to f.d + wt(f,w)
      add w to the frontier
    else if the path to w via f is shorter:
      update w's shortest path length
```

f: 3

Settled set: {4, 0, 1, 2, 3}

Frontier set: {} Empty => done!



shortest-paths(4)

Dijkstra's Shortest Paths: Pseudocode

```
S = { }; F = {v};  v.d = 0;          Initialize Settled to empty
while  (F ≠ {})  {                   Initialize Frontier to the start node
   f = node in F with min d value;
   Remove f from F, add it to S;
   for each neighbor w of f {
      if (w not in S or F) {
         w.d =  f.d + weight(f, w);
         add w to F;
      } else if (f.d+weight(f,w) < w.d) {
         w.d = f.d+weight(f,w);
      }
   }
}
```

Dijkstra's Shortest Paths: Pseudocode

```
S = { }; F = {v};  v.d = 0;              Initialize Settled to empty
while  (F ≠ {})  {                       Initialize Frontier to the start node
   f = node in F with min d value;       While the frontier isn't empty:
   Remove f from F, add it to S;            move node f with smallest d
   for each neighbor w of f {                from F to S
      if (w not in S or F) {
          w.d =  f.d + weight(f, w);
          add w to F;
      } else if (f.d+weight(f,w) < w.d) {
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Dijkstra's Shortest Paths: Pseudocode

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   f = node in F with min d value;         While the frontier isn't empty:
   Remove f from F, add it to S;             move node f with smallest d
                                               from F to S
   for each neighbor w of f {              For each neighbor w of f:
      if (w not in S or F) {                 if we've never seen w before:
         w.d =  f.d + weight(f, w);            set its path length
         add w to F;                           add it to frontier
      } else if (f.d+weight(f,w) < w.d) {
         w.d = f.d+weight(f,w);
      }
   }
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```

Dijkstra's Shortest Paths: Pseudocode

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         w.d = f.d+weight(f,w);
      }
   }
}
```

```
Initialize Settled to empty
Initialize Frontier to the start node
While the frontier isn't empty:
   move node f with smallest d
    from F to S
 For each neighbor w of f:
   if we've never seen w before:
     set its path length
     add it to frontier
   else if path to w via f is shorter:
       update w's shortest path length
```

What if we want to know the shortest path?

```
S = { }; F = {v};  v.d = 0;
while  (F ≠ {})  {
   f = node in F with min d value;
   Remove f from F, add it to S;
   for each neighbor w of f {
      if (w not in S or F) {
         w.d =  f.d + weight(f, w);
         add w to F;
      } else if (f.d+weight(f,w) < w.d) {
         w.d = f.d+weight(f,w);
      }
   }
}
```

- At termination: for each reachable node n, n.d stores the **length** of the shortest path from v to n.
- We didn't keep track of **how** to get from v to n!

What if we want to know the shortest path?

```
S = { }; F = {v};  v.d = 0; v.bp = null;
while  (F ≠ {})  {
   f = node in F with min d value;
   Remove f from F, add it to S;
   for each neighbor w of f {
      if (w not in S or F) {
         w.d =  f.d + weight(f, w);
         w.bp = f;
         add w to F;
      } else if (f.d+weight(f,w) < w.d) {
         w.d = f.d+weight(f,w);
         w.bp = f
      }
   }
}
```

Each node could store the full path, but that would be expensive to keep updated.

Strategy: maintain a **backpointer** at each node pointing to the previous node in the shortest path.

What if we want to know the shortest path? Example

```
S = { }; F = {v};  v.d = 0; v.bp = null;
while  (F ≠ {})  {
   f = node in F with min d value;
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   for each neighbor w of f {
      if (w not in S or F) {
         w.d =  f.d + weight(f, w);
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         add w to F;
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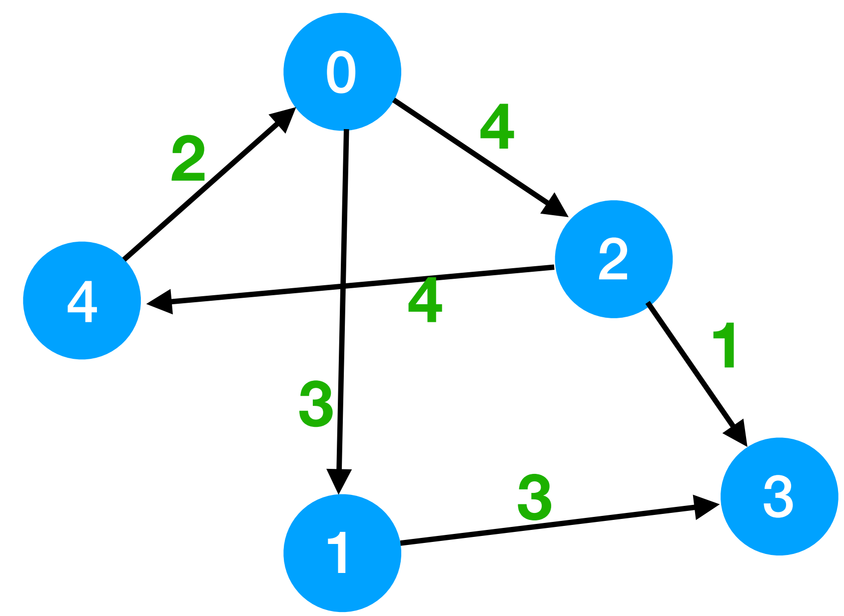
```



`shortest-paths(4)`

Strategy: maintain a **backpointer** at each node pointing to the previous node in the shortest path.

Let’s Dijkstra

- Half get the algorithm, half get the graphs. Pair up with someone with the other thing.
- Run the algorithm on each graph:
  - (first) 5-node graph: start at node S
  - (second) Other graph: start at node D


The next slide is really important.

Implementing Dijkstra Efficiently (A4)

```
S = { }; F = {v};  v.d = 0; v.bp = null;
while  (F ≠ {})  {
   f = node in F with min d value;
   Remove f from F, add it to S;
  for each neighbor w of f {
     if (w not in S or F) {
         w.d =  f.d + weight(f, w);
         w.bp = f;
         add w to F;
    } else if (f.d+weight(f,w) < w.d) {
         w.d = f.d+weight(f,w);
         w.bp = f
    }
  }
}
```

1. Store Frontier in a min-heap priority queue with d-values as priorities.
2. To efficiently iterate over neighbors, use an adjacency list graph representation.
3. Could store w.d and w.bp in Node class; in A4, we use a HashMap<Node,PathData>
4. No need to explicitly store Settled or Unexplored sets:
   a node is in S or F iff it is in the map.

Implementing Dijkstra Efficiently (A4)

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S = { }; F = {v};  v.d = 0; v.bp = null;
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   for each neighbor w of f {
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         w.d =  f.d + weight(f, w);
         w.bp = f;
         add w to F;
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         w.d = f.d+weight(f,w);
         w.bp = f
      }
   }
}
```

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Implementing Dijkstra Efficiently (A4)

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while  (F ≠ {})  {
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         w.d =  f.d + weight(f, w);
         w.bp = f;
         add w to F;
      } else if (f.d+weight(f,w) < w.d) {
         w.d = f.d+weight(f,w);
         w.bp = f
      }
   }
}
```

1. Store Frontier in a min-heap priority queue with d-values as priorities.
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Implementing Dijkstra Efficiently (A4)

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   Remove f from F, add it to S;
   for each neighbor w of f {
      if (w not in S or F) {
         w.d =  f.d + weight(f, w);
         w.bp = f;
         add w to F;
      } else if (f.d+weight(f,w) < w.d) {
         w.d = f.d+weight(f,w);
         w.bp = f
      }
   }
}
```

1. Store Frontier in a min-heap priority queue with d-values as priorities.
2. To efficiently iterate over neighbors, use an adjacency list graph representation.
3. **Could store w.d and w.bp in Node class; in A4, we use a HashMap<Node,PathData>**
4. No need to explicitly store Settled or Unexplored sets:
   a node is in S or F iff it is in the map.

Implementing Dijkstra Efficiently (A4)

```
S = { }; F = {v};  v.d = 0; v.bp = null;
while  (F ≠ {})  {
   f = node in F with min d value;
   Remove f from F, add it to S;
   for each neighbor w of f {
      if (w not in S or F) {
         w.d =  f.d + weight(f, w);
         w.bp = f;
         add w to F;
      } else if (f.d+weight(f,w) < w.d) {
         w.d = f.d+weight(f,w);
         w.bp = f
      }
   }
}
```

1. Store Frontier in a min-heap priority queue with d-values as priorities.
2. To efficiently iterate over neighbors, use an adjacency list graph representation.
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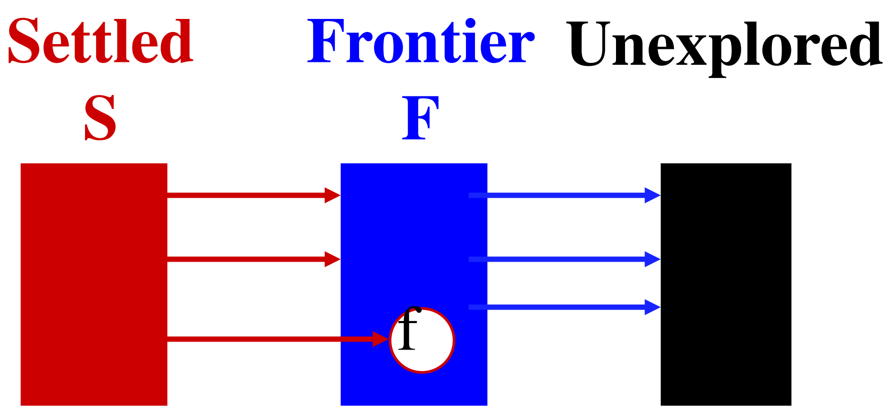
4. **No need to explicitly store Settled or Unexplored sets: a node is in S or F iff it is in the map.**

Proof of Correctness

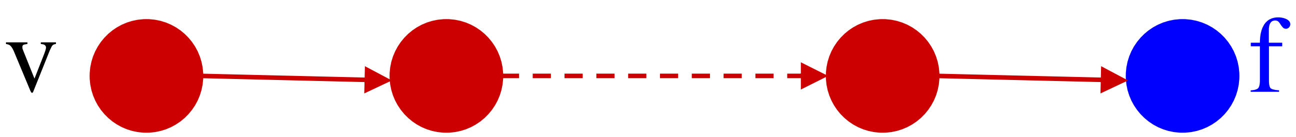
- Dijkstra's algorithm is **greedy**: it makes a sequence of *locally* optimal moves, which results in the *globally* optimal solution.
  - Most algorithms don't work like this - need to prove that it results in the global optimum.
- Specifically: It is not obvious that there cannot still be a shorter path to the Frontier node with smallest d-value.

Proof of Correctness: Invariant



The while loop in Dijkstra's algorithm maintains a 3-part invariant:

1. For a Settled node s, a shortest path from v to s contains only settled nodes and s.d is length of shortest v -> s path.



2. For a Frontier node f, at least one v -> f path contains only settled nodes (except perhaps for f) and f.d is the length of the shortest such path

3. All edges leaving S go to F (or: no edges from S to Unexplored)

Proof of Correctness: Theorem

```
S = { }; F = {v};  v.d = 0;
while  (F ≠ {})  {
  f = node in F with min d value;
  Remove f from F, add it to S;
  for each neighbor w of f {
    if (w not in S or F) {
      w.d =  f.d + weight(f, w);
      add w to F;
    } else if (f.d+weight(f,w) < w.d) {
      w.d = f.d+weight(f,w);
    }
  }
}
```

Theorem: For a node f in the Frontier with minimum d value (over all nodes in the Frontier), f.d is the shortest-path distance from v to f.

Proof: Show that any other path from v to if has length >= f.d

Case 1: if v is in F, then S is empty and v.d = 0, which is trivially the shortest distance from v to v.

Proof of Correctness: Theorem

```
S = { }; F = {v};  v.d = 0;
while  (F ≠ {})  {
   f = node in F with min d value;
   Remove f from F, add it to S;
   for each neighbor w of f {
      if (w not in S or F) {
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```
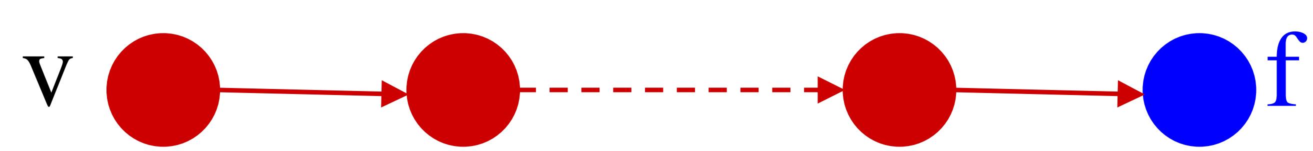

Theorem: For a node f in the Frontier with minimum d value (over all nodes in the Frontier), f.d is the shortest-path distance from v to f.

Proof: Show that any other path from v to if has length >= f.d

Case 2: v is in S. Part 2 of the invariant says:

- f.d is the length of the shortest path from v to f containing all settled nodes except f, and f.d is the length of such a path.



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   f = node in F with min d value;
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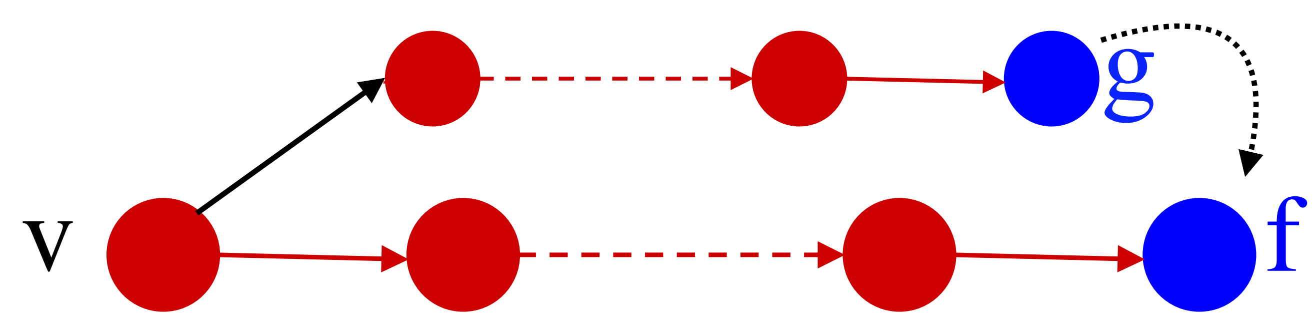
Case 2: v is in S. Part 2 of the invariant says:

- f.d is the length of the shortest path from v to f containing all settled nodes except f, and f.d is the length of such a path.

Any other v-f path must either be longer or go through another frontier node g then arrive at f:



Proof of Correctness: Theorem

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while  (F ≠ {})  {
   f = node in F with min d value;
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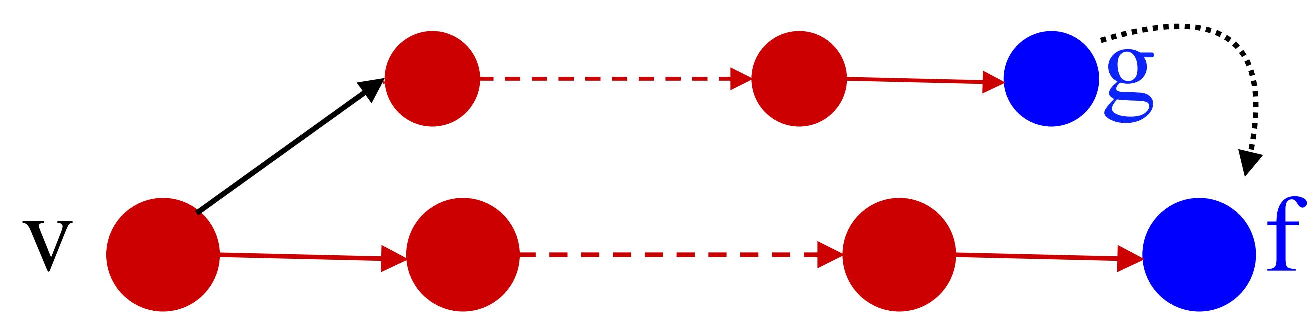
Case 2: v is in S. Part 2 of the invariant says:

- f.d is the length of the shortest path from v to f containing all settled nodes except f, and f.d is the length of such a path.

Any other v-f path must either be longer or go through another frontier node g then arrive at f:

d.f <= d.g,
so that path cannot be shorter



Proof of Correctness: Invariant Maintenance

```
S = { }; F = {v};  v.d = 0;
while  (F ≠ {})  {
   f = node in F with min d value;
   Remove f from F, add it to S;
   for each neighbor w of f {
      if (w not in S or F) {
         w.d =  f.d + weight(f, w);
         add w to F;
      } else if (f.d+weight(f,w) < w.d) {
         w.d = f.d+weight(f,w);
      }
   }
}
```

1. For a Settled node s, a shortest path from v to s contains only settled nodes and s.d is length of shortest v -> s path.
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   Remove f from F, add it to S;
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         w.d =  f.d + weight(f, w);
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         w.d = f.d+weight(f,w);
      }
   }
}
```

1. For a Settled node s, a shortest path from v to s contains only settled nodes and s.d is length of shortest v -> s path.
2. For a Frontier node f, at least one v -> f path contains only settled nodes (except perhaps for f) and f.d is the length of the shortest such path
3. All edges leaving S go to F (or: no edges from S to Unexplored)

At initialization:

1. S is empty; trivially true.
2. v.d = 0, which is the shortest path.
3. S is empty, so no edges leave it.

Proof of Correctness: Invariant Maintenance

```
S = { }; F = {v};  v.d = 0;
while  (F ≠ {})  {
   f = node in F with min d value;
   Remove f from F, add it to S;
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1. For a Settled node s, a shortest path from v to s contains only settled nodes and s.d is length of shortest v -> s path.
2. For a Frontier node f, at least one v -> f path contains only settled nodes (except perhaps for f) and f.d is the length of the shortest such path
3. All edges leaving S go to F (or: no edges from S to Unexplored)

At each iteration:

1. Theorem says f.d is the shortest path, so it can safely move to S
2. Updating w.d maintains Part 2 of the invariant.
3. Each neighbor is either already in F or gets moved there.