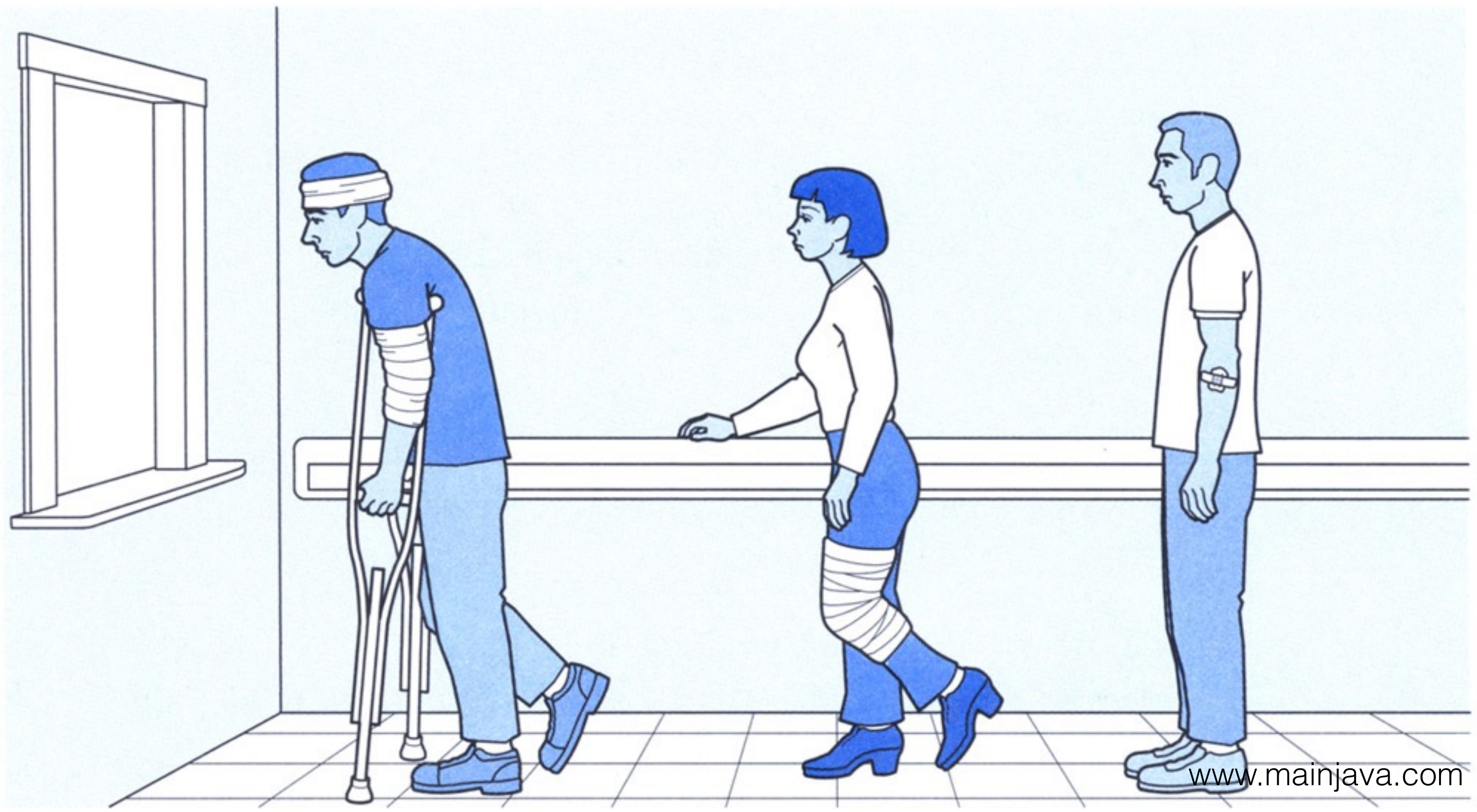

CSCI 241

Lecture 13:
One last bit of AVL
Priority Queue
Heaps

Announcements

- Welcome back! It’s been a while.
- A2 is due Tuesday night:
  - Look through the supplemental AVL slides!
  - Visit me in office hours!
  - Email me if you’re stuck over the weekend!
  - I’ll post an FAQ if I get many duplicate questions.
- Midterm exam is next Friday.
  - Covers material through today.
  - Study guide will be available by Monday.
  - One double-sided sheet of hand-written notes is allowed.
  - Quizzes, ABCDs, etc. are the most efficient study tool.

Goals

- Understand some efficiency gotchas:
  - A1 - copying the array in merge
  - A2 - computing height in rebalance
- Understand the purpose and interface of the Priority Queue ADT.
- Know the definition and properties of a heap.
- Know how heaps are stored in practice.
- Know how to implement add, peek, and poll heap operations.

A1 Efficiency Gotcha

```
merge(A, start, mid, end):
  B = deep copy of A
  i = start
  j = mid
  k = 0
  while i < mid and j < end:
    if B[i] < B[j]:
      A[k] = B[i]
      i++
    else:
      A[k] = B[j]
      j++
    k++


  while i < mid:
    A[k] = B[i]
    i++, k++

  while j < end:
    A[k] = B[j]
    j++, k++
```

A1 Efficiency Gotcha

```
merge(A, start, mid, end):
  B = deep copy of A
  i = start
  j = mid
  k = 0
  while i < mid and j < end:
    if B[i] < B[j]:
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      i++
    else:
      A[k] = B[j]
      j++
    k++

  while i < mid:
    A[k] = B[i]
    i++, k++

  while j < end:
    A[k] = B[j]
    j++, k++
```

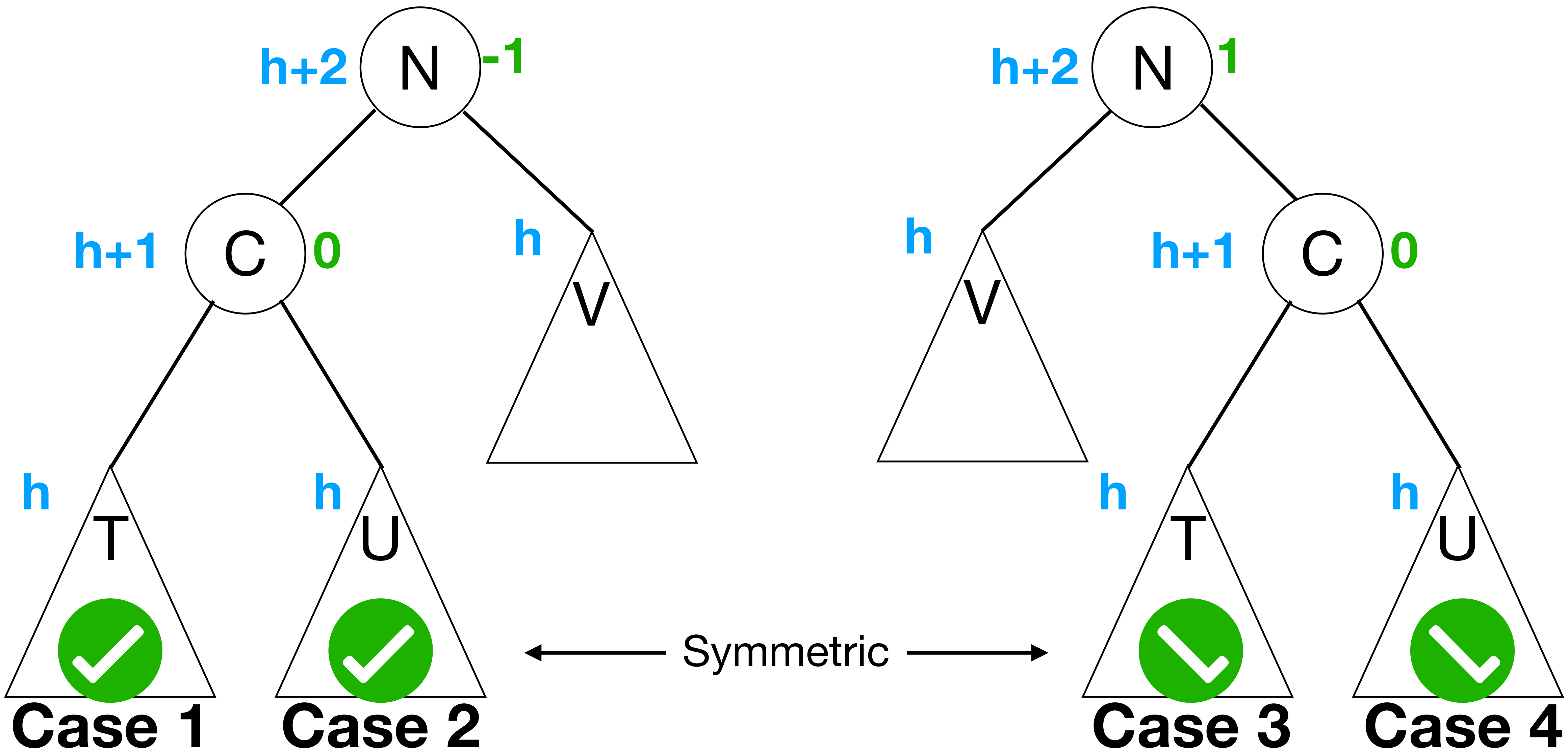
- In merge(), copying the entire array A makes it O(A.length)
- Our runtime analysis relied on merge being O(A[end-start])
- Copying **all** of A instead of the relevant **range** of A makes mergesort O(n^2)!

AVL Insertion

```
/* insert a node with value v into the
 * tree rooted at n. pre: n is not null. */
insert(Node n, int v):
  if n.value == v: return // (duplicate)
  if v < n.value:
    if n has left:
      insert(n.left, v)
    else:
      // attach new node w/ value v to n.left
  else: // v > n.value
    if n has right:
      insert(n.right, v)
    else:
      // attach new node w/ value v to n.right
  rebalance(n);
```

AVL Rebalance

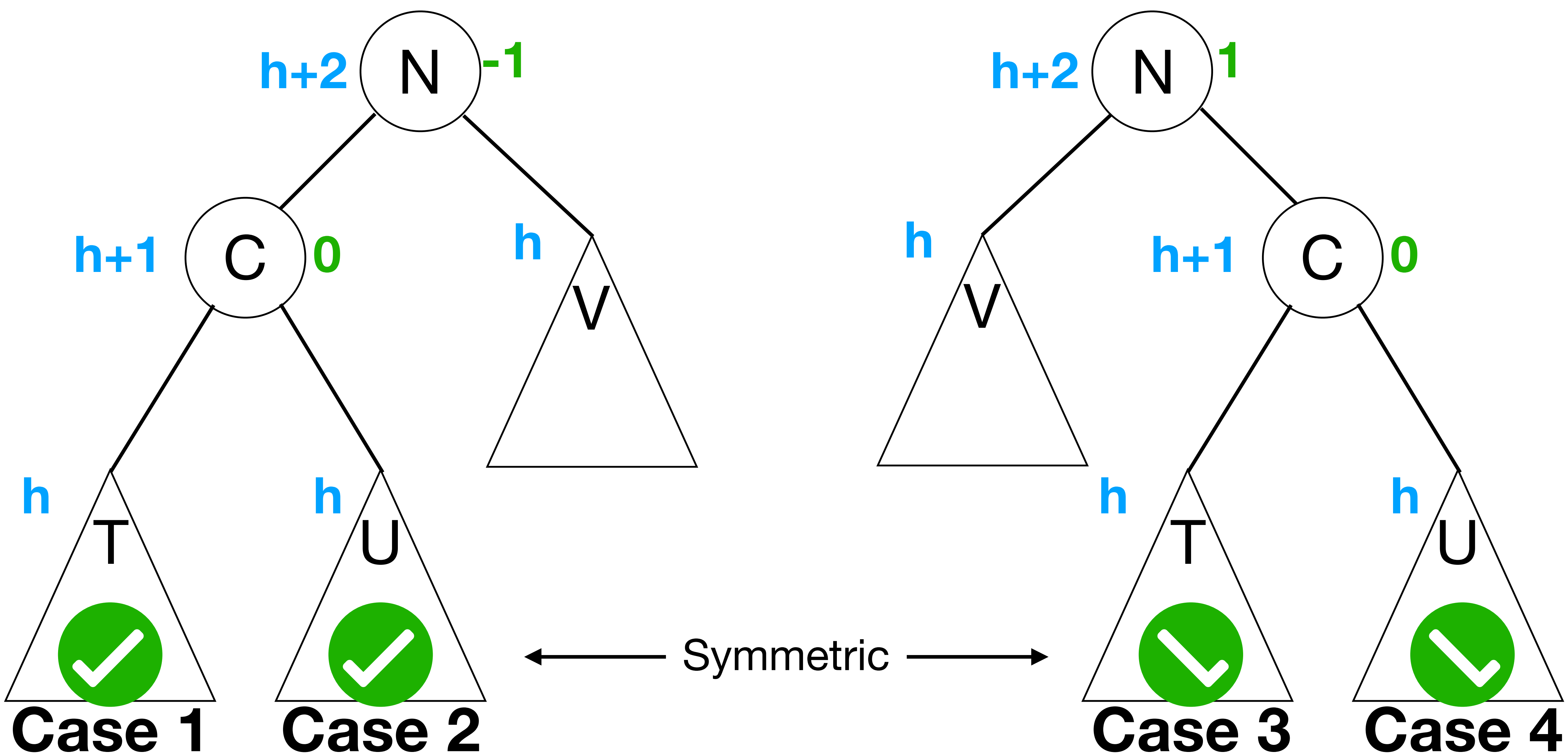
Read the supplemental AVL slides. Do the exercises.
Convince yourself that the code for rebalance does the right thing.



An insertion that unbalances n could gone one of four places.

AVL Rebalance

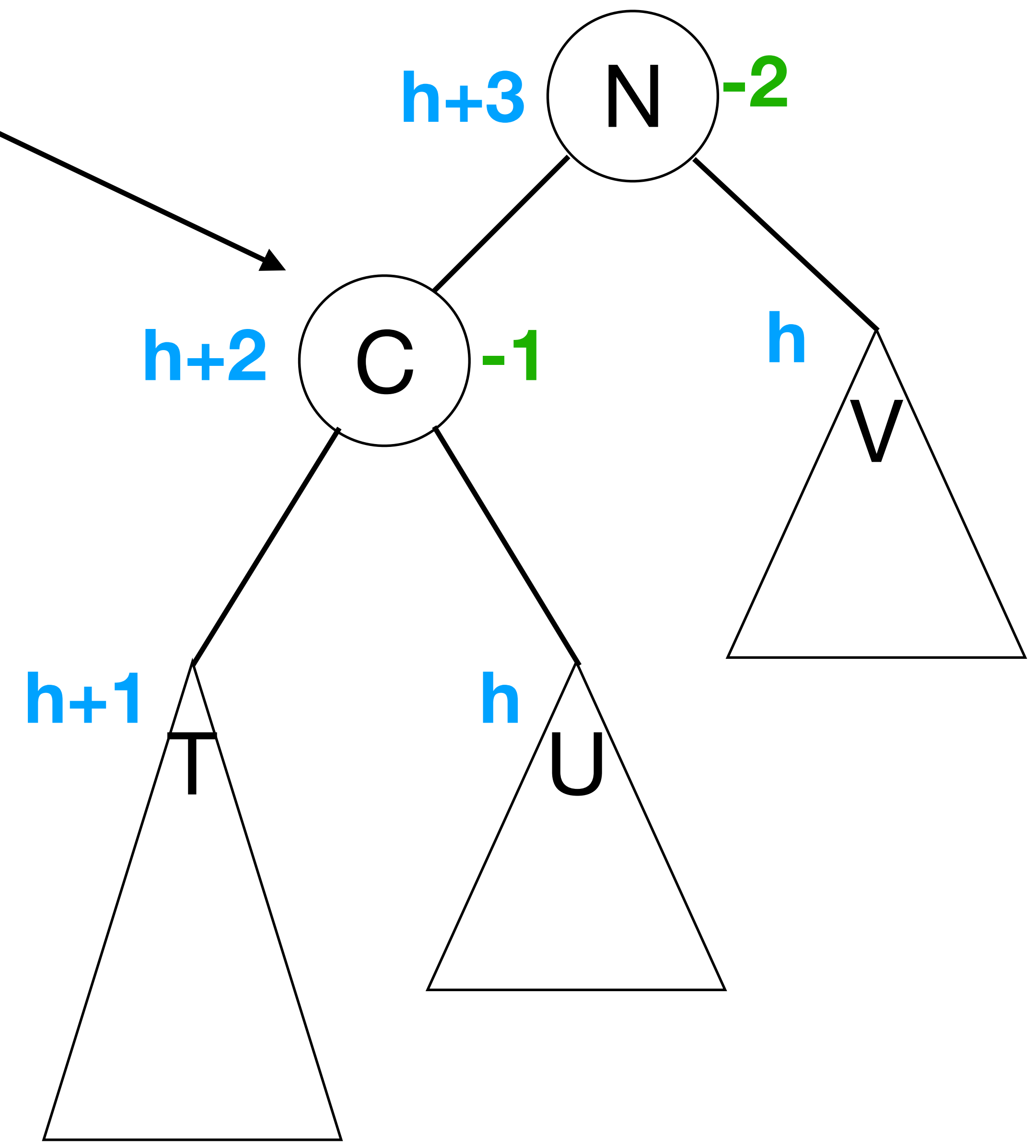
Before an insertion that unbalances n,
the tree must look like one of these:



An insertion that unbalances n could gone one of four places.

Implementation

```
void rebalance(n):
  if bal(n) < -1:
    if bal(n.left) < 0
      // case 1:
      // rightRot(n)
    else:
      // case 2:
      // leftRot(n.L);
      // rightRot(n)
  else if bal(n) > 1:
    if bal(n.right) < 0:
      // case 3:
      // rightRot(n.R);
      // leftRot(n)
    else:
      // case 4:
      // leftRot(n)
```



Case 1

Implementation

```
void rebalance(n):
  if bal(n) < -1:
    if bal(n.left) < 0
      // case 1:
      // rightRot(n)
    else:
      // case 2:
      // leftRot(n.L);
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  else if bal(n) > 1:
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      // case 3:
      // rightRot(n.R);
      // leftRot(n)
    else:
      // case 4:
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```
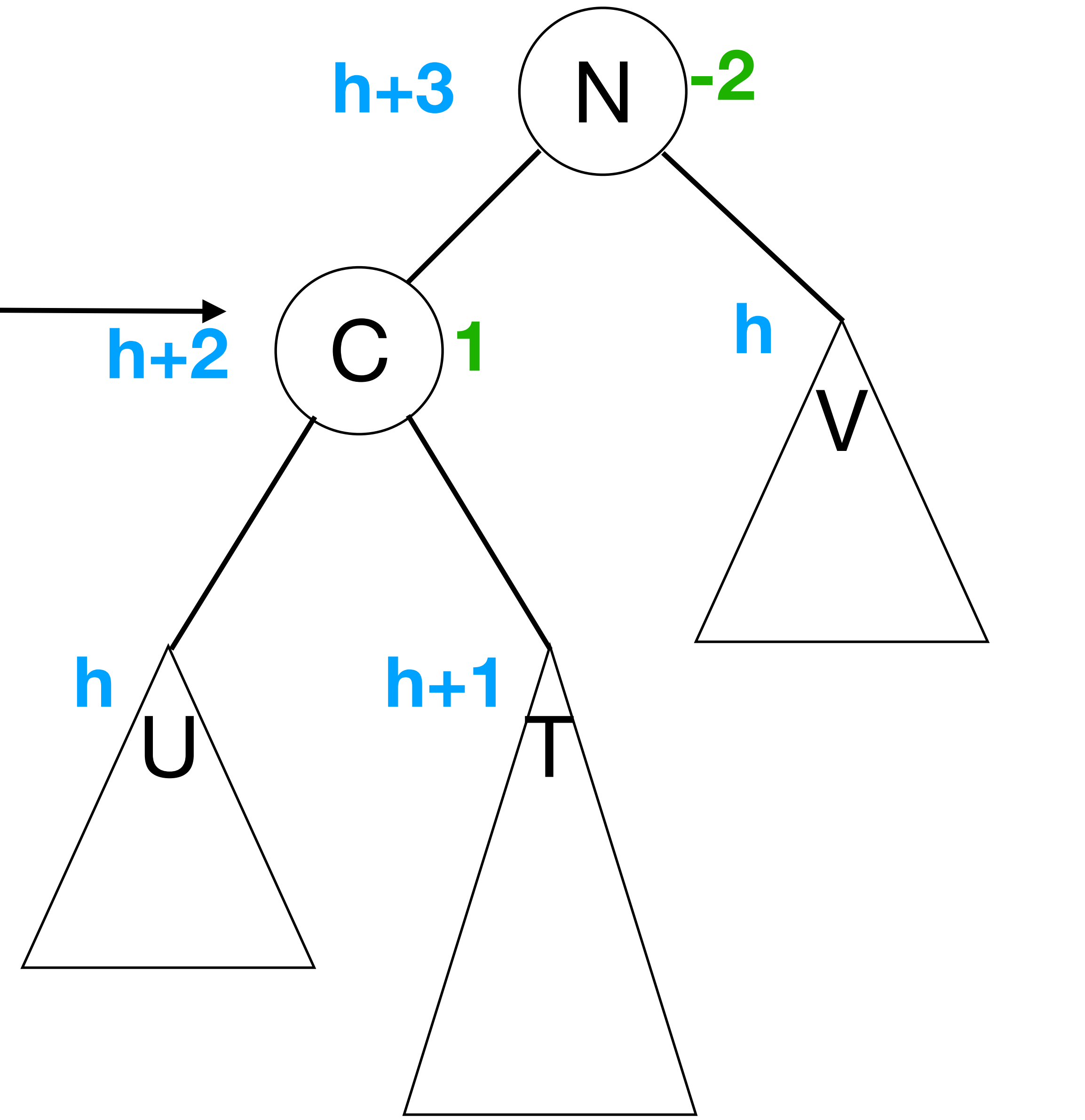
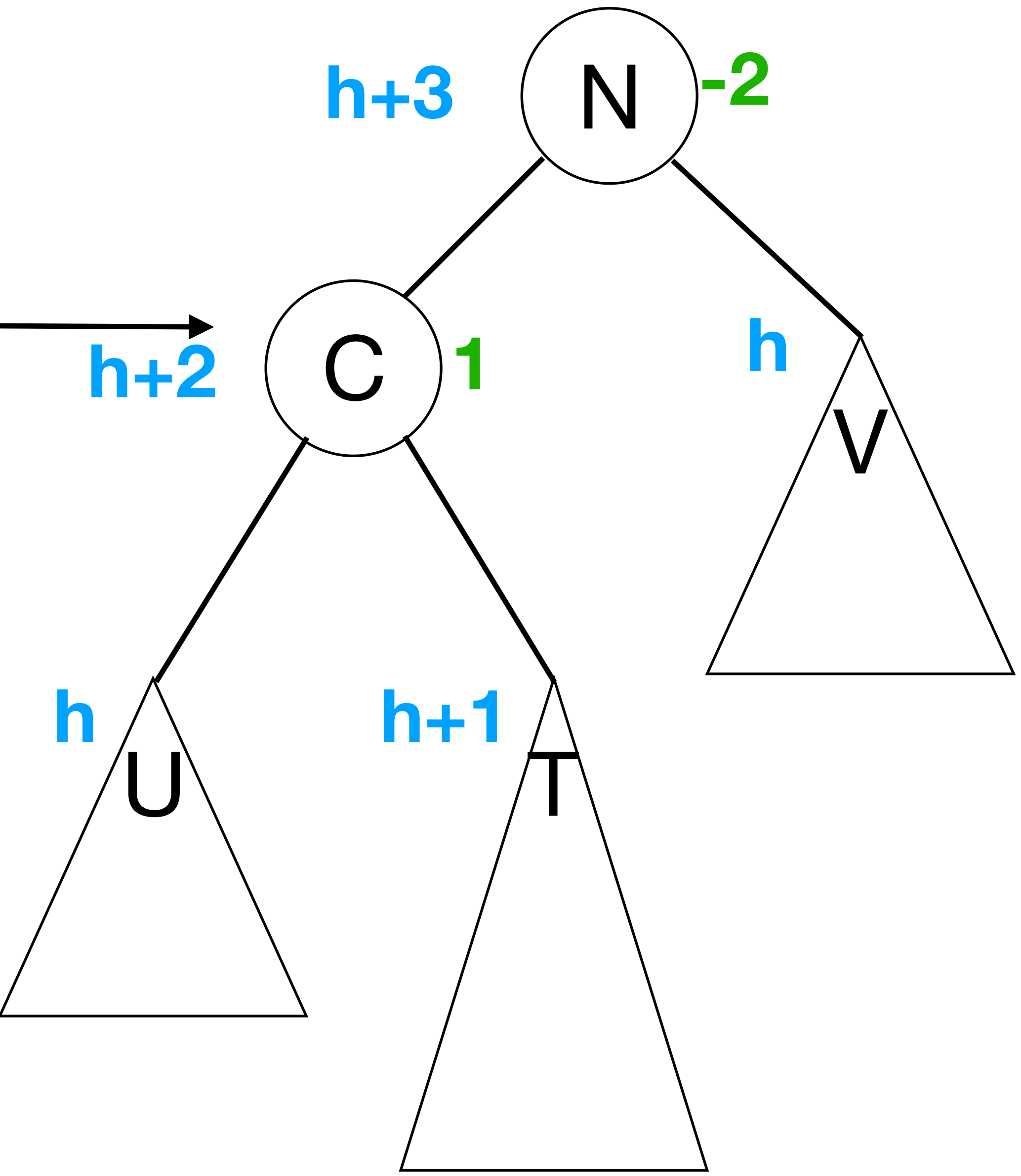



Case 2

Implementation

```
void rebalance(n):
  if bal(n) < -1:
    if bal(n.left) < 0
      // case 1:
      // rightRot(n)
    else:
      // case 2:
      // leftRot(n.L);
      // rightRot(n)
  else if bal(n) > 1:
    if bal(n.right) < 0:
      // case 3:
      // rightRot(n.R);
      // leftRot(n)
    else:
      // case 4:
      // leftRot(n)
```



Case 2

Cases 3 and 4 are symmetric with 2 and 1

Implementation

```
void rebalance(n):
  if bal(n) < -1:
    if bal(n.left) < 0
      // case 1:
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      // rightRot(n.R);
      // leftRot(n)
    else:
      // case 4:
      // leftRot(n)
```

Cases 3 and 4 are symmetric with 2 and 1.

Maintaining Height

```
void rebalance(n):
  if bal(n) < -1:
    if bal(n.left) < 0
      // case 1:
      // rightRot(n)
    else:
      // case 2:
      // leftRot(n.L);
      // rightRot(n)
  else if bal(n) > 1:
    if bal(n.right) < 0:
      // case 3:
      // rightRot(n.R);
      // leftRot(n)
    else:
      // case 4:
      // leftRot(n)
```

- `bal` needs to know childrens' heights.
- Computing height by recursively walking the tree (as in lab3) is O(n)!
- Doing this every time we compute `bal` would ruin our O(log n) runtime of AVL insertion!
- Each node needs to keep track of its height.

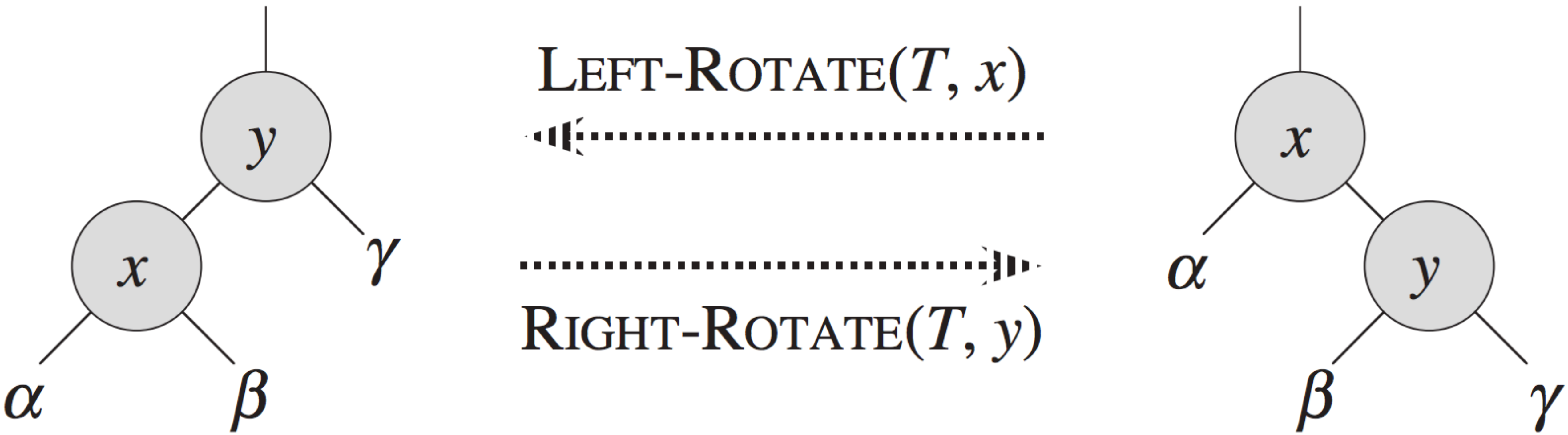
Maintaining Height

- Each node needs to keep track of its height.
  - Key idea: if my childrens' heights are correct, my height can be calculated in constant time:
    - `n.height = 1 + max(n.left.height, n.right.height)`
  - When can a node's height change?
    - After a rotation
    - After an insertion

Maintaining Height

- Each node needs to keep track of its height.
    - Key idea: if my childrens' heights are correct, my height can be calculated in constant time:
        - `n.height = 1 + max(n.left.height, n.right.height)`
    - When can a node's height change?
        - **After a rotation**
        - After an insertion

Height after rotations

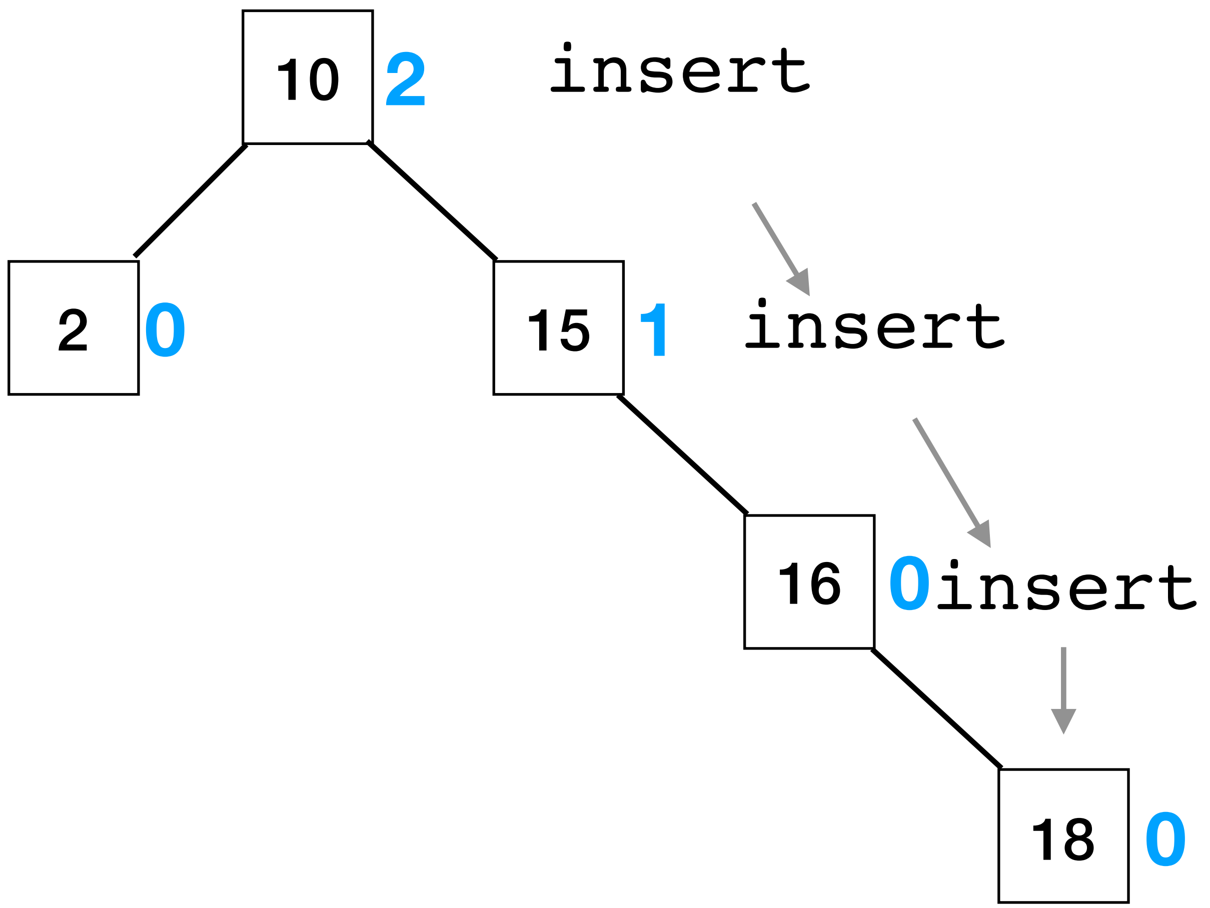


- Heights of child subtrees (alpha, beta, gamma) can't change!
- Heights of x and y change, but can be calculated **directly from (already-correct) heights of children**.

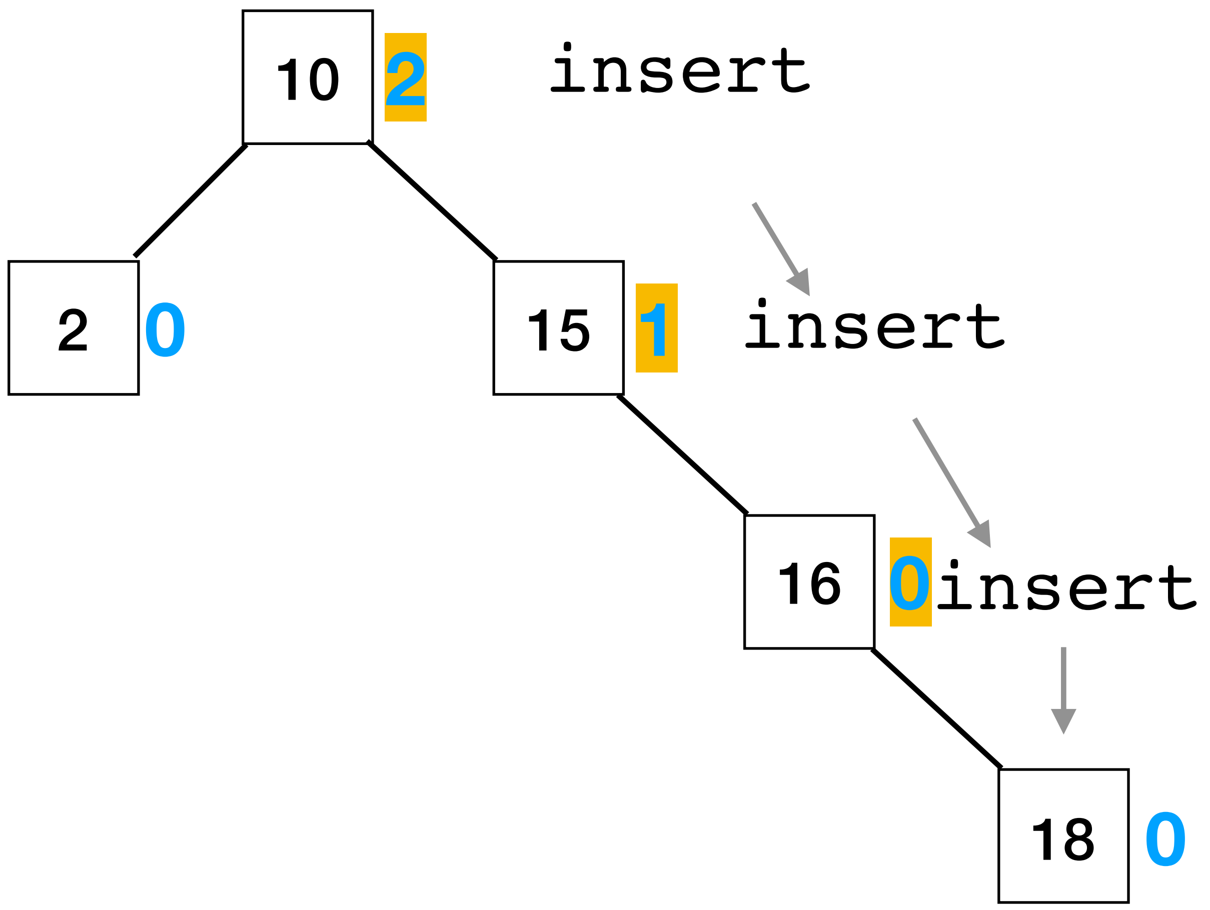
Maintaining Height

- Each node needs to keep track of its height.
  - Key idea: if my childrens' heights are correct, my height can be calculated in constant time:
    - `n.height = 1 + max(n.left.height, n.right.height)`
  - When can a node's height change?
    - After a rotation
    - **After an insertion**

Height after insertion

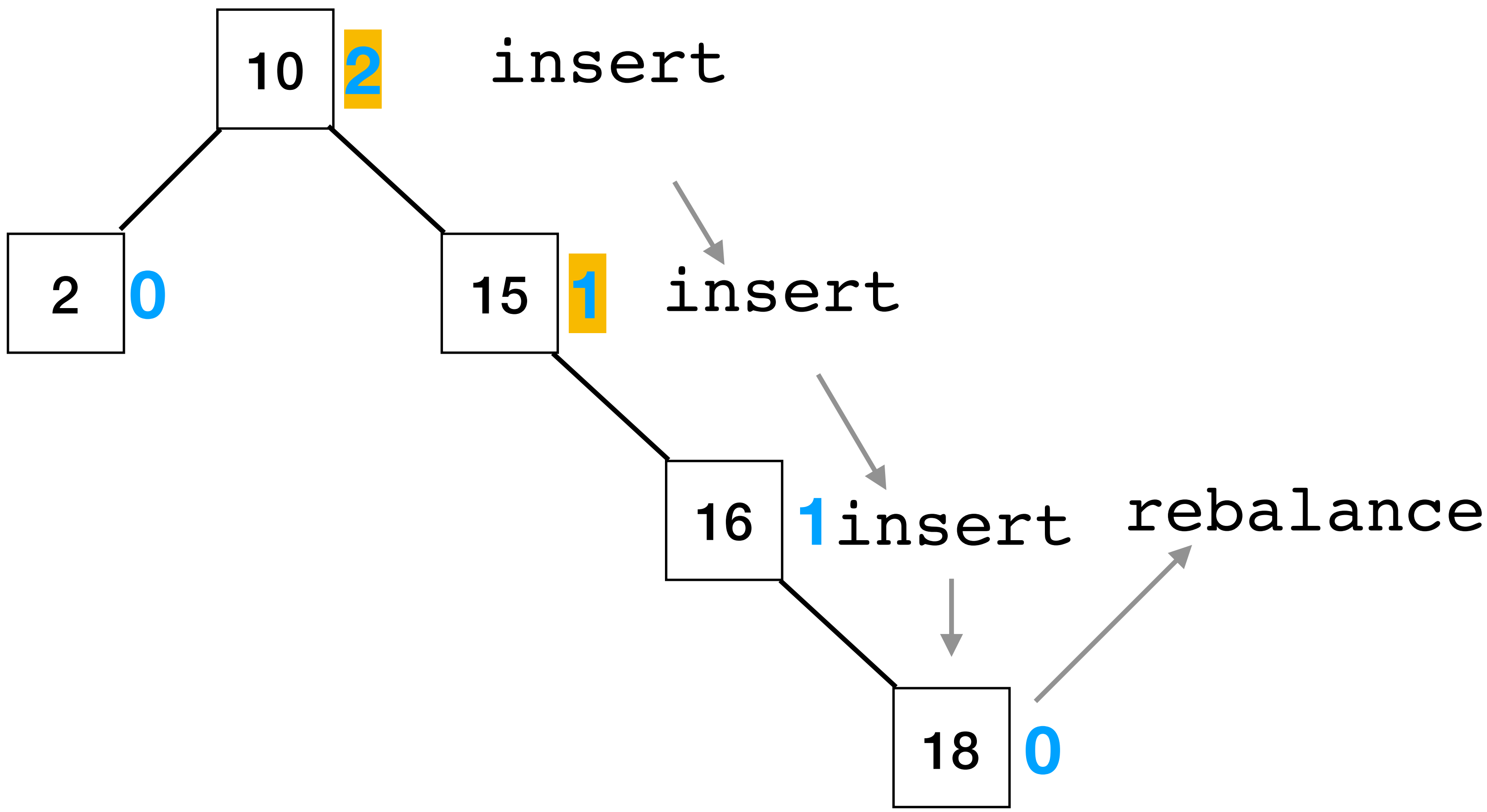


Height after insertion



- All nodes on the path to root need to have height updated.
- We’re calling rebalance on *exactly* those nodes!
- Each node’s height update can safely be computed from the child heights.

Height after insertion



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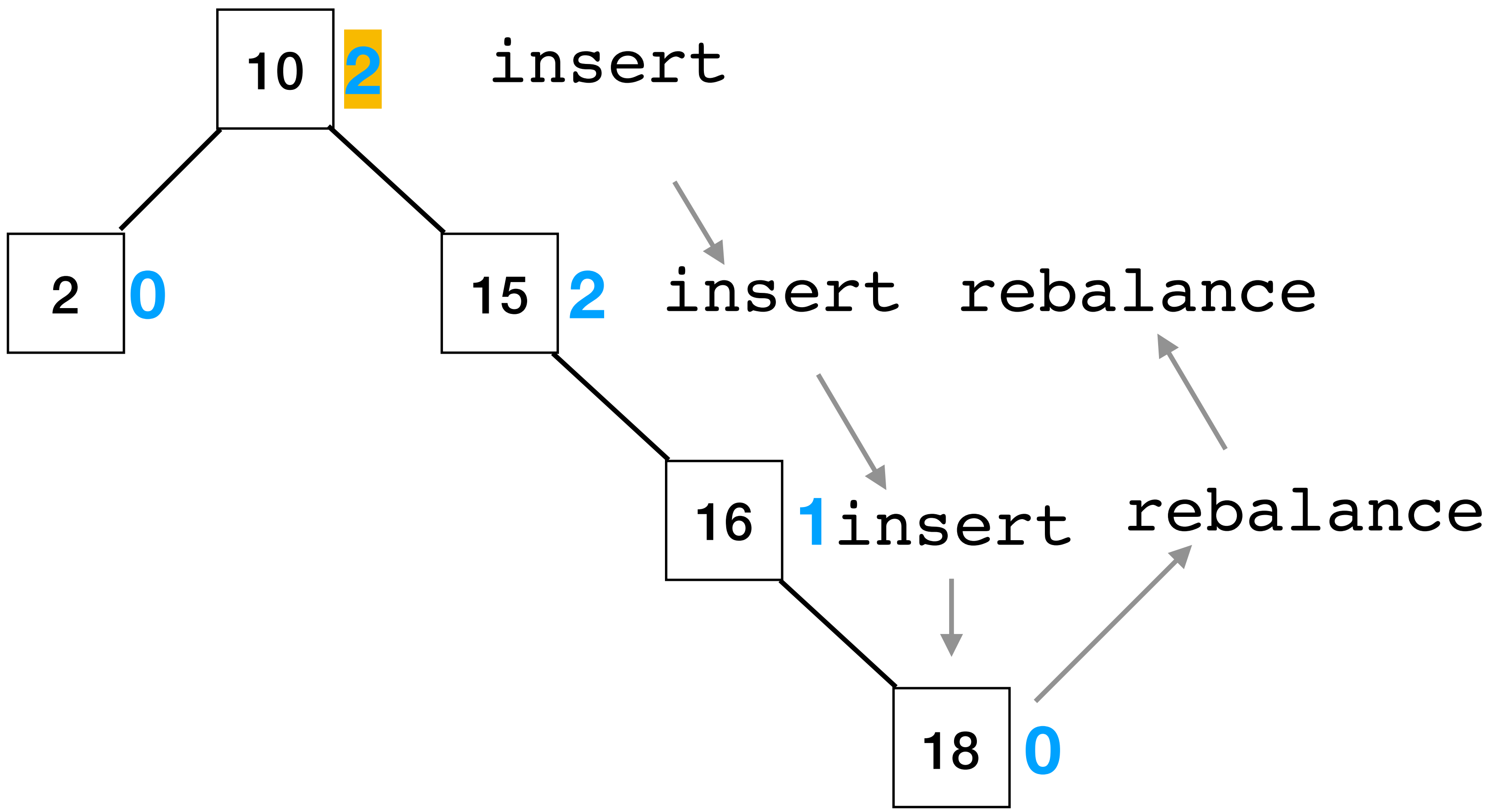
Height after insertion



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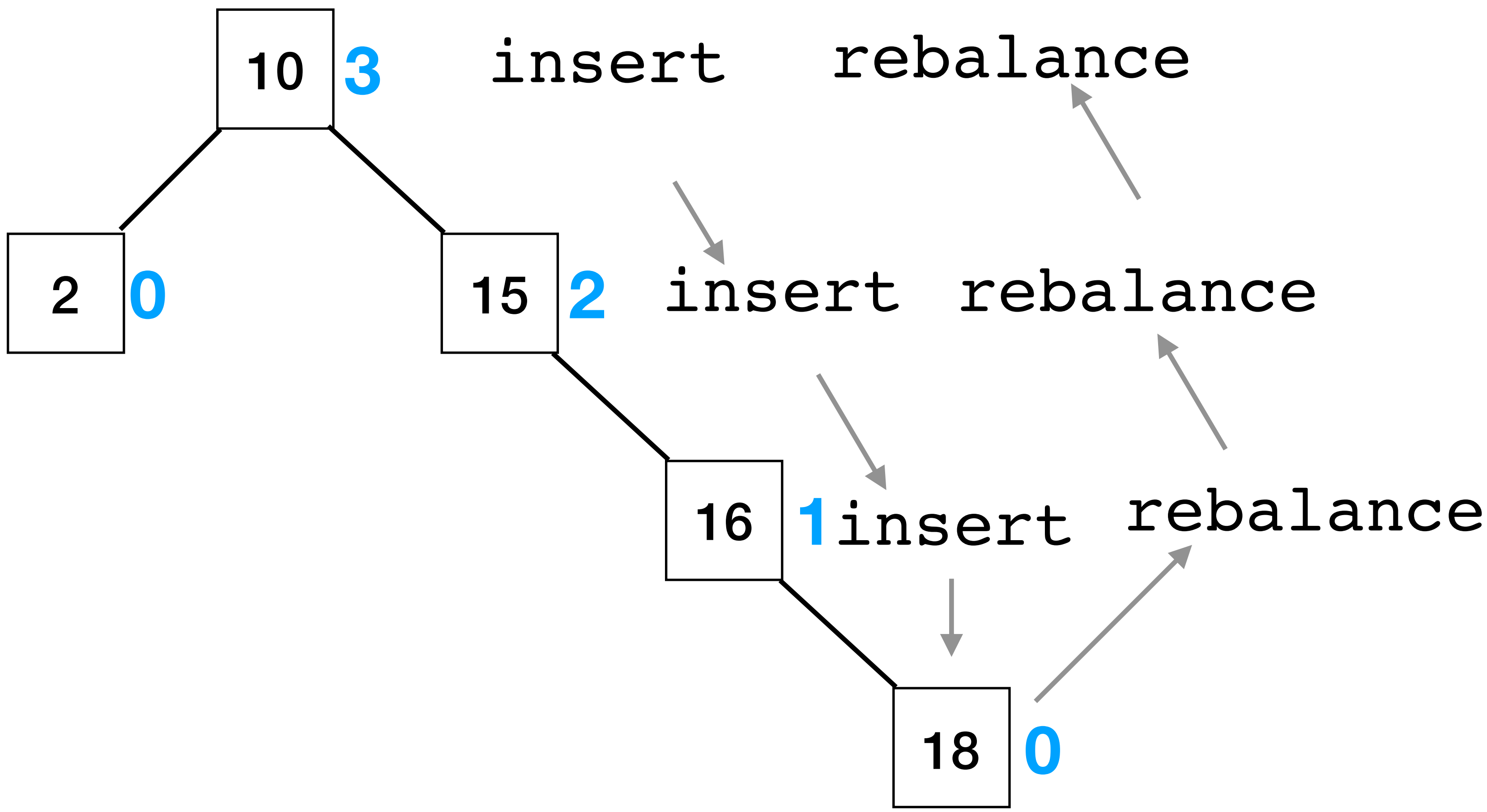
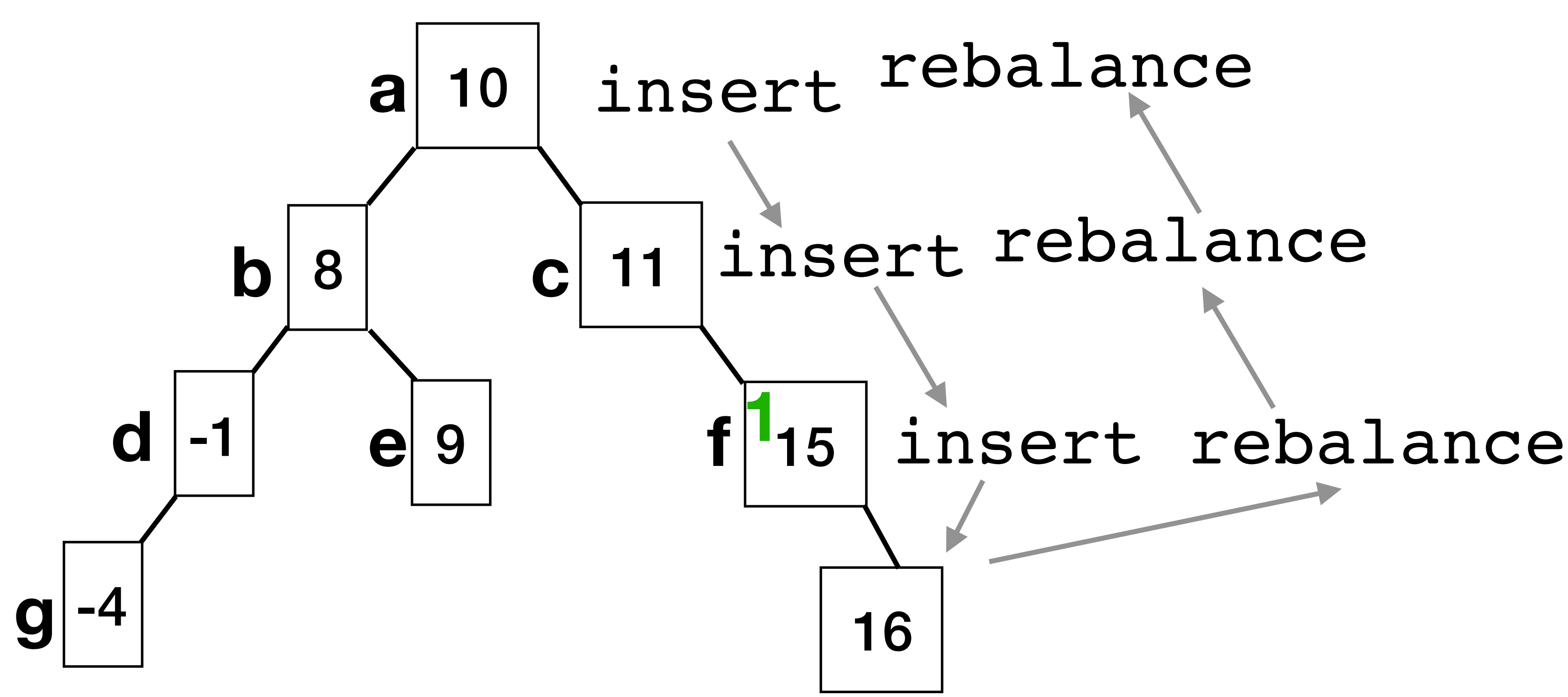
Height after insertion



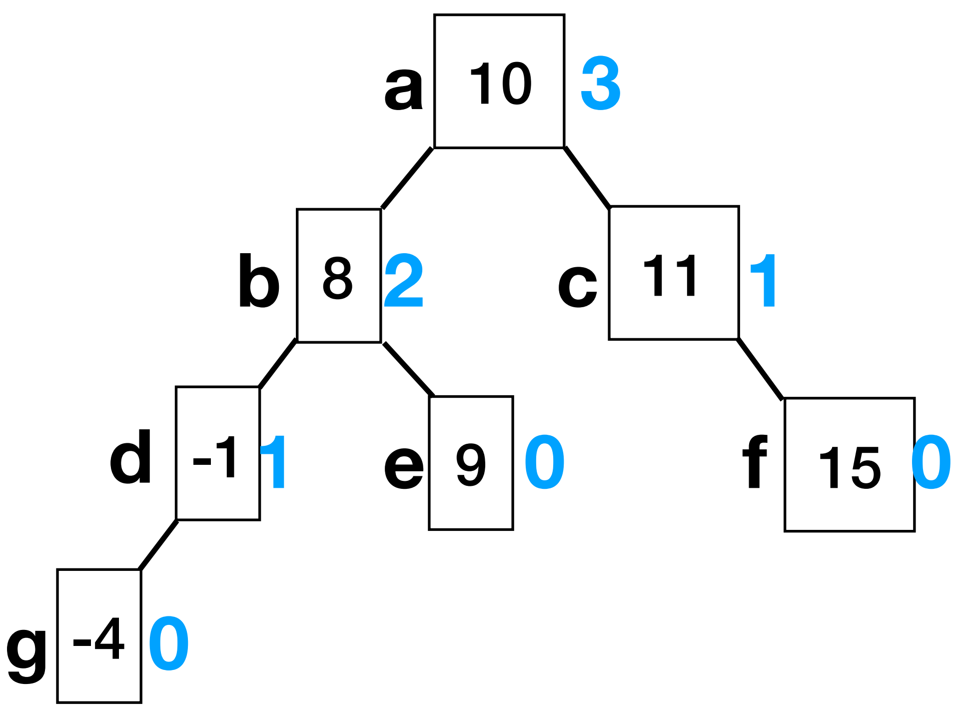
- All nodes on the path to root need to have height updated.
- We're calling rebalance on *exactly* those nodes!
- The calls happen bottom-up, so each child heights have been updated already.

Height of AVL Trees

- As usual, runtime of search, insert, and remove are all O(height).
  - A rotation is O(1), so even if we have to rebalance every node on the path to the root, it's still only h*O(1) rebalances.



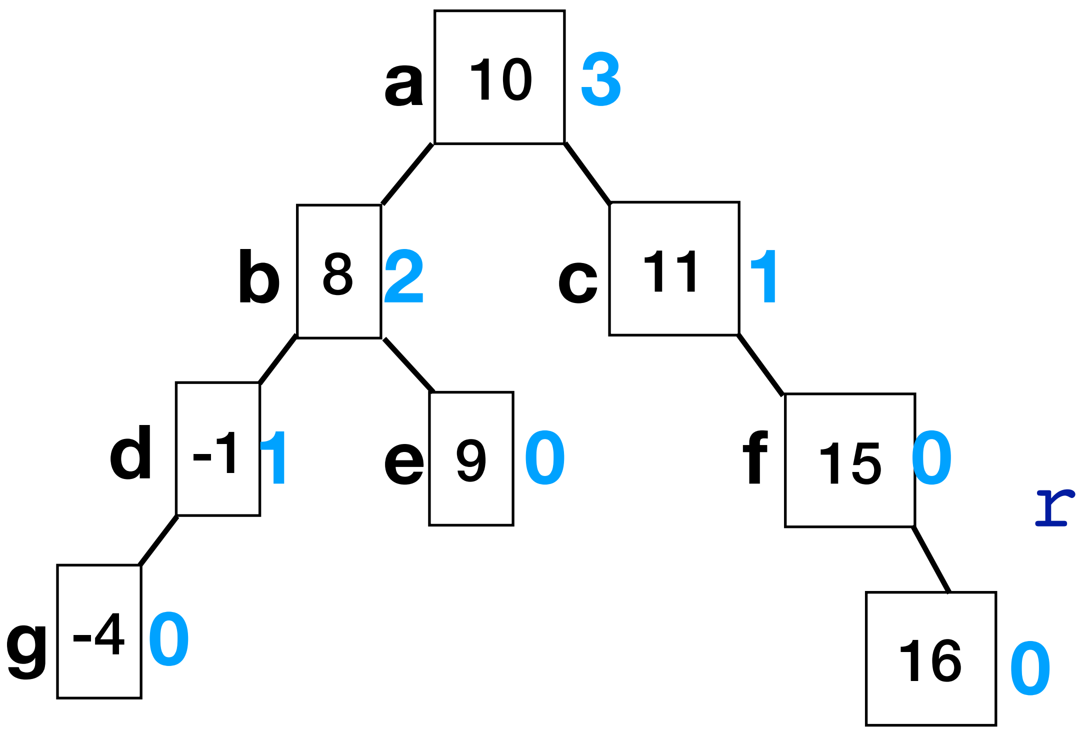
Height Updates during Insertion
(another visualization for your reference)




```
insert(Node n, int v):
  //…(other case, irrelevant here)
  else: // v > n.value
    if n has right:
      insert(n.right, v)
   else:
      // attach new node w/ value
      //    v to n.right
  rebalance(n);
```

```
insert(a, 16)
=>insert(c, 16)
  =>insert(f, 16)
    =>attach new node
      rebalance(f)
    rebalance(c)
  rebalance(a)
```

AVL Insertion

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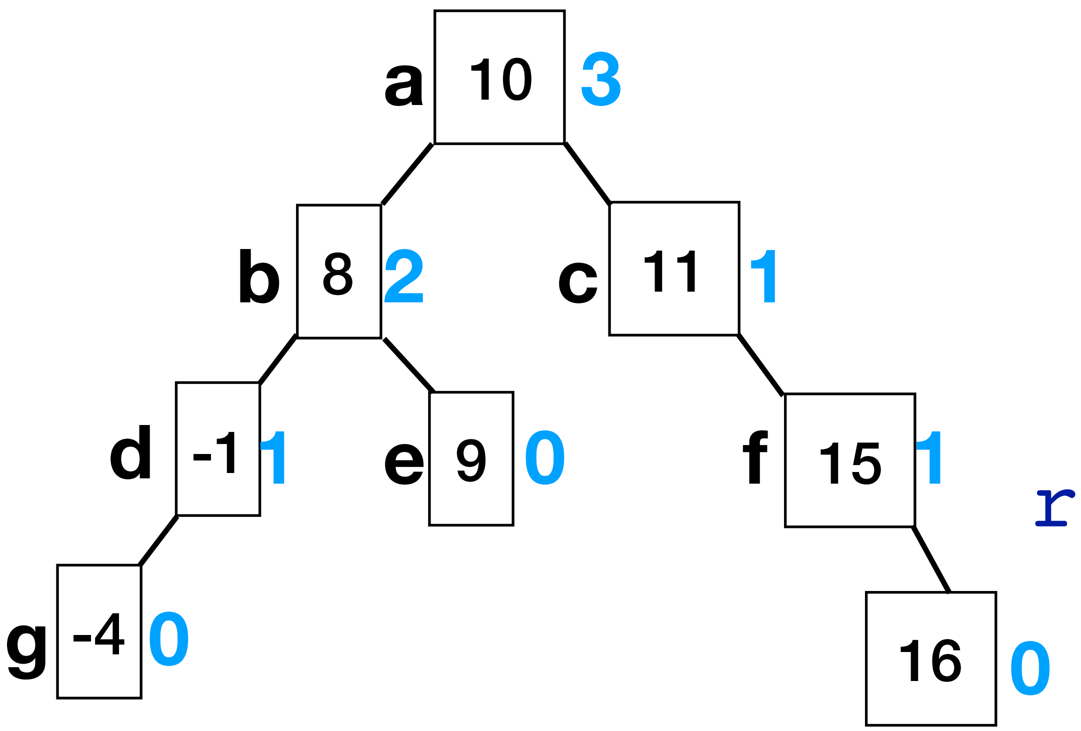
AVL Insertion

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      rebalance(f)
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  rebalance(a)
```

update f's height

AVL Insertion

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  //…(other case, irrelevant here)
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      //    v to n.right
  rebalance(n);
```
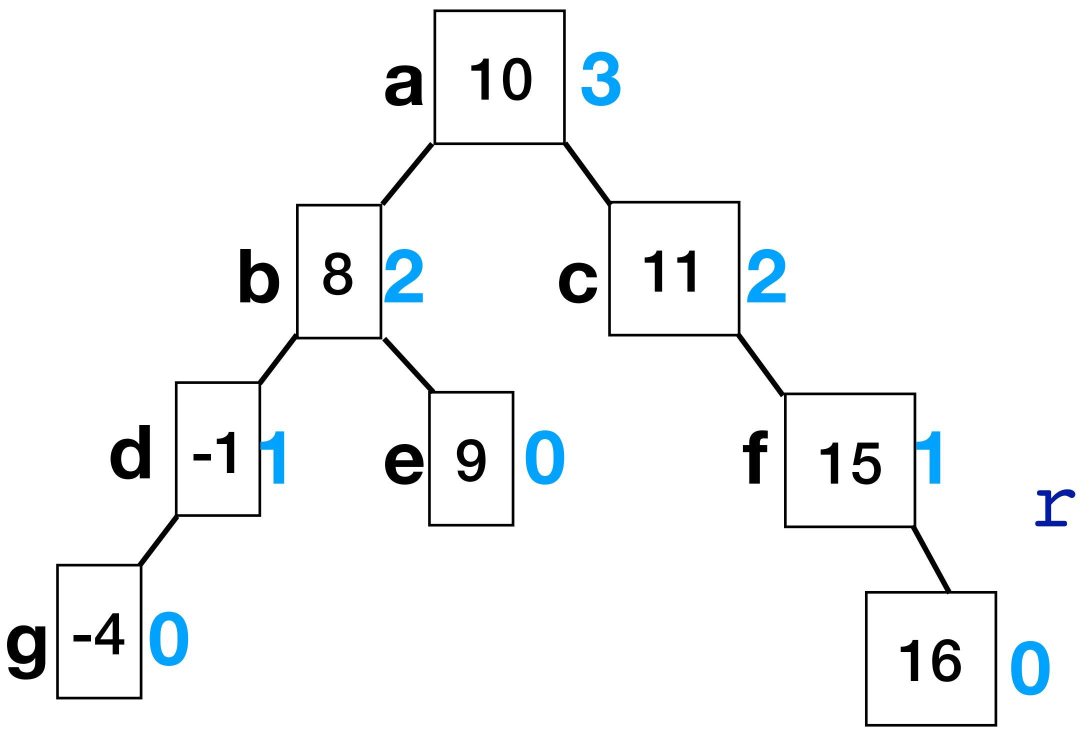
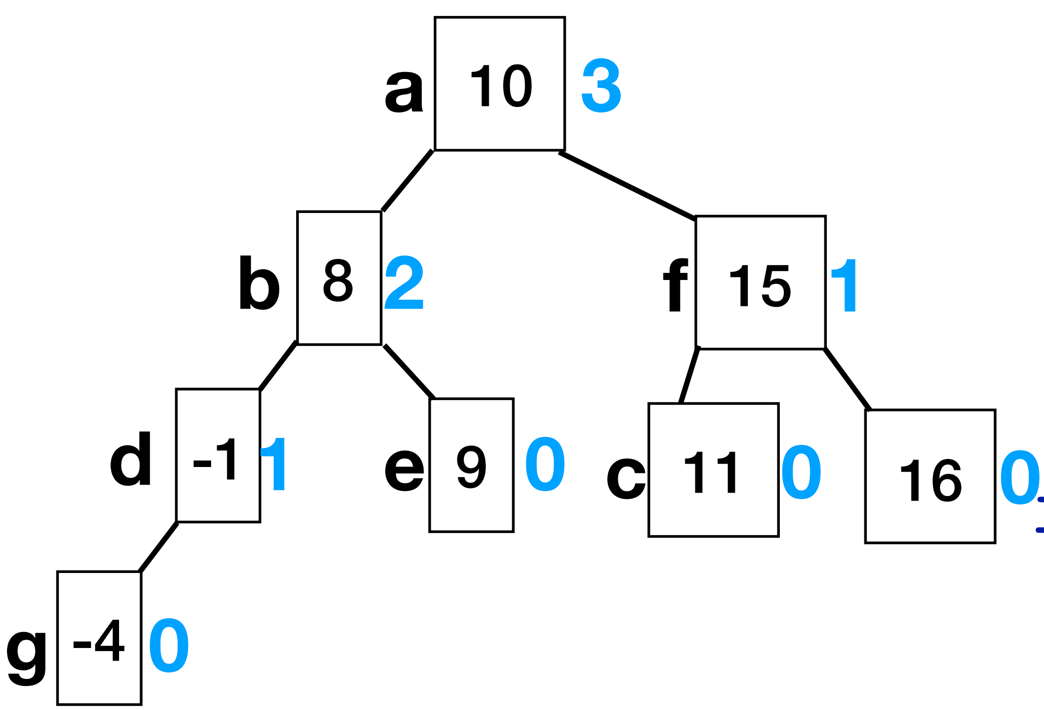




```
insert(a, 16)
=>insert(c, 16)
  =>insert(f, 16)
    =>attach new node
      rebalance(f)
    rebalance(c)
  rebalance(a)
```

update f's height

update c's height

AVL Insertion

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rebalance(n);
```

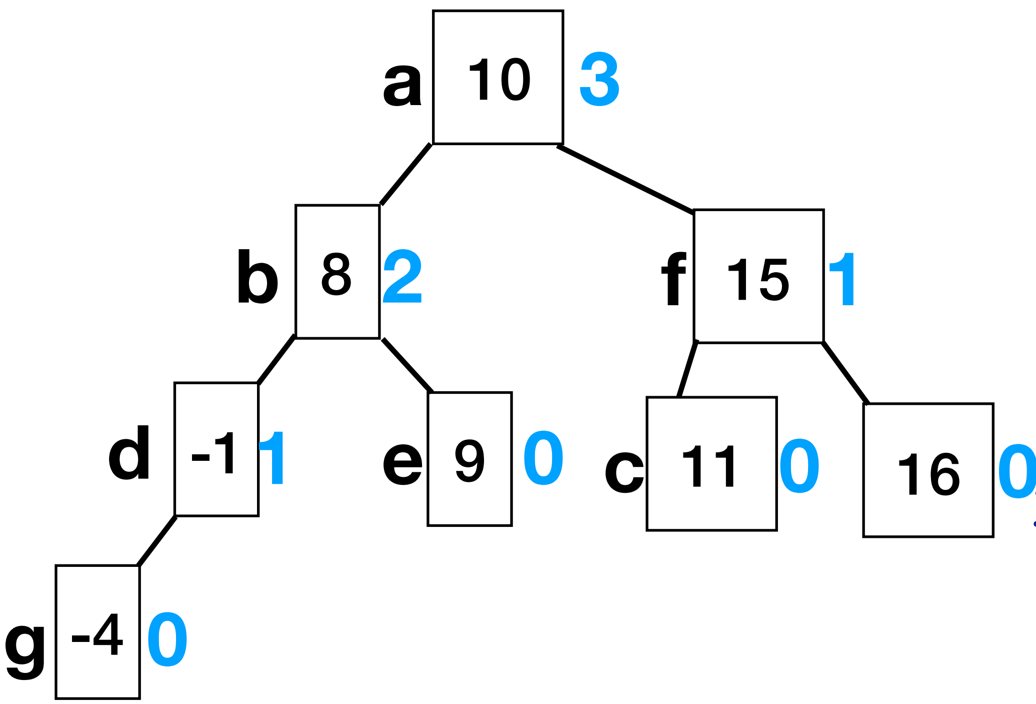
```
insert(a, 16)
=>insert(c, 16)
  =>insert(f, 16)
    =>attach new node
      rebalance(f)
    rebalance(c)
  rebalance(a)
```

update f's height

update c's height

(rotation happens)

AVL Insertion

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    if n has right:
      insert(n.right, v)
   else:
      // attach new node w/ value
      //   v to n.right
rebalance(n);
```

```
insert(a, 16)
=>insert(c, 16)
  =>insert(f, 16)
    =>attach new node
      rebalance(f)        update f's height
    rebalance(c)        update c's height
  rebalance(a)        update a's height
```

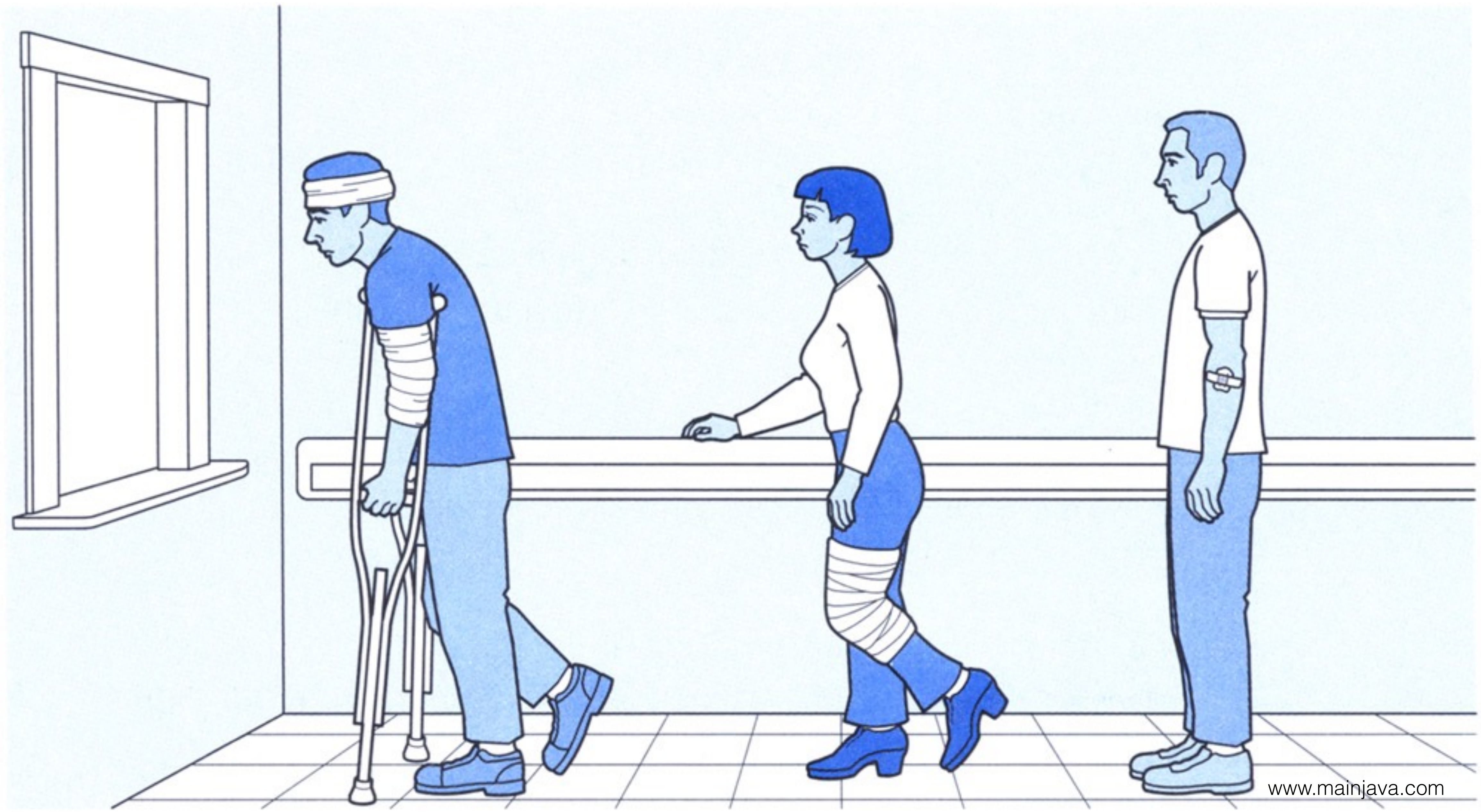
Height of AVL Trees

- As usual, runtime of search, insert, and remove are all O(height)
- What is the worst-case height of an AVL tree with n nodes?
  - Exact proof is CSCI 405 material; uses the fibonacci sequence(!)
  - Spoiler: the answer is **O(log n)**
- Intuition: To add to root's height, you have to add to height of every subtree in one of root's subtrees.

Removing from AVL Tree

- Much like insertion: remove as usual, rebalance as necessary at each level up to the root.
- Whereas insertion only ever requires only one rebalance, deletion can require many
  - but still no more than the tree’s height.

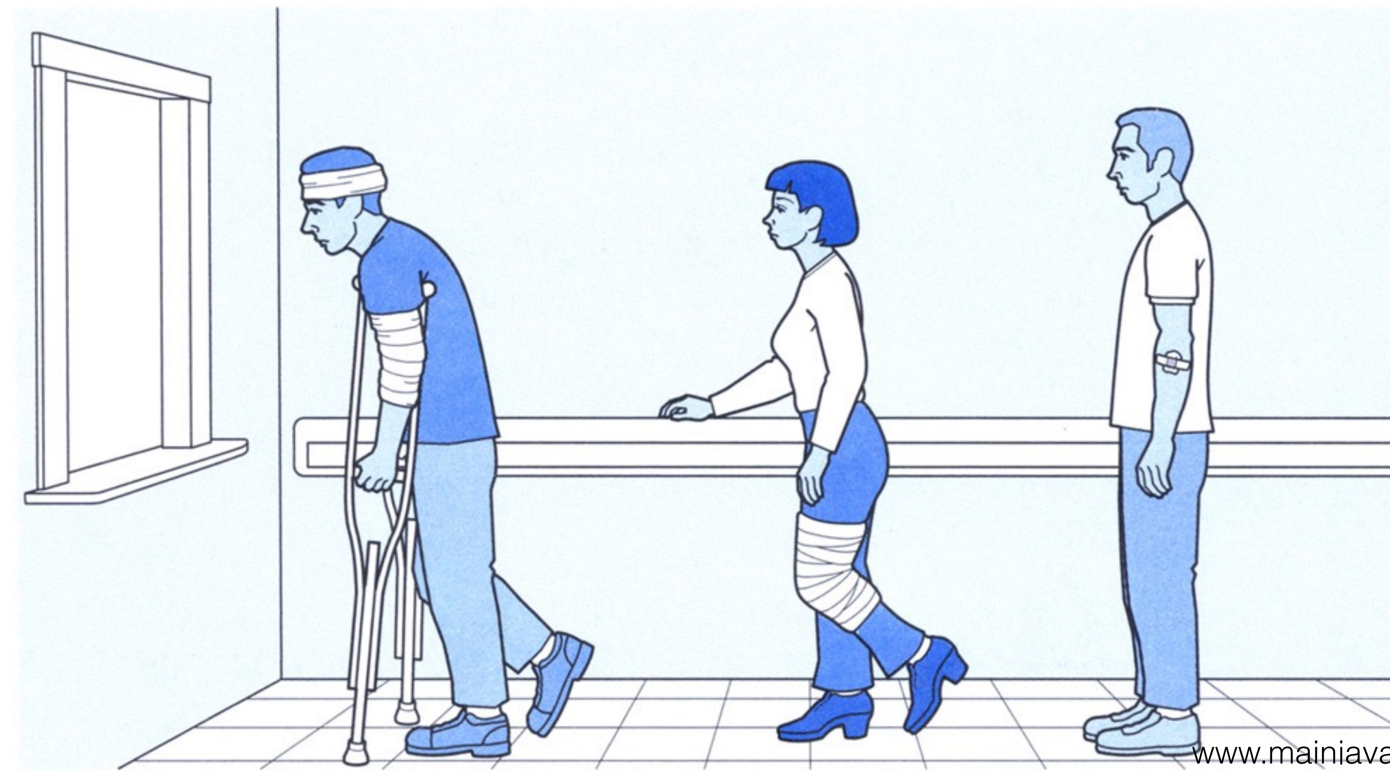
Priority Queues



Queue vs Priority Queue

add (enqueue):
inserts an item into the queue

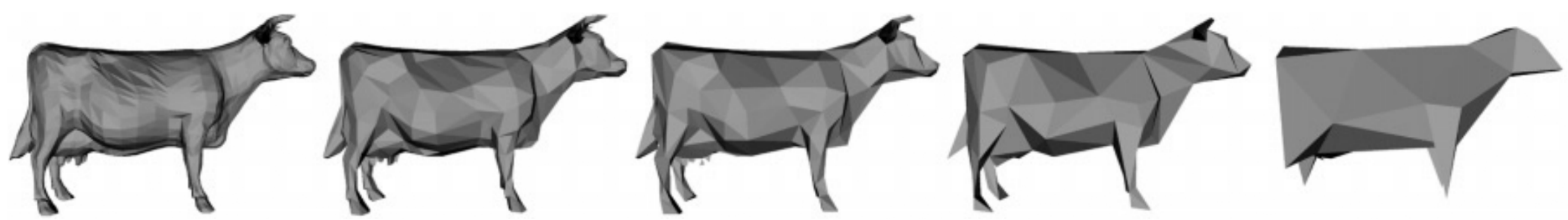
remove (dequeue):
removes the first item to be inserted (FIFO)



add (enqueue):
inserts an item into the queue

remove (poll):
remove the **highest-priority** item from the queue

Uses for Priority Queues



- Computer Graphics: mesh simplification
- Graph algorithms: shortest paths, spanning trees
- Statistics: maintain largest M values in a sequence
- Graphics and simulation: "next time of contact" for colliding bodies
- AI Path Planning: A* search (e.g., Map directions)
- Operating systems: load balancing, interrupt handling
- Discrete optimization: bin packing, scheduling

Priority Queues

Like a Queue, but:

- Each item in the queue has an associated **priority** which is some type that implements **Comparable**
- **remove()** returns item with the "highest priority"
  - or, the element with the "smallest" associated priority value
  - Ties are broken arbitrarily

```
interface PriorityQueue<E> {
 boolean add(E e); // insert e
 E peek(); // return min element
 E poll(); // remove/return min element
 void clear();
 boolean contains(E e);
 boolean remove(E e);
 int size();
 Iterator<E> iterator();
}
```

Priority Queue: LinkedList implementation

An unordered list:

- **add()** - new element at front of list - O(1)
- **poll()** - requires searching the list - O(n)
- **peek()** - requires searching the list - O(n)

An ordered list:

- **add()** - requires searching the list - O(n)
- **poll()** - min element is kept at front - O(1)
- **peek()** - min element is kept at front - O(1)

Question to ponder:

What would be the runtime of add, peek, and poll if you implement a Priority Queue using a BST?

What about an AVL tree?

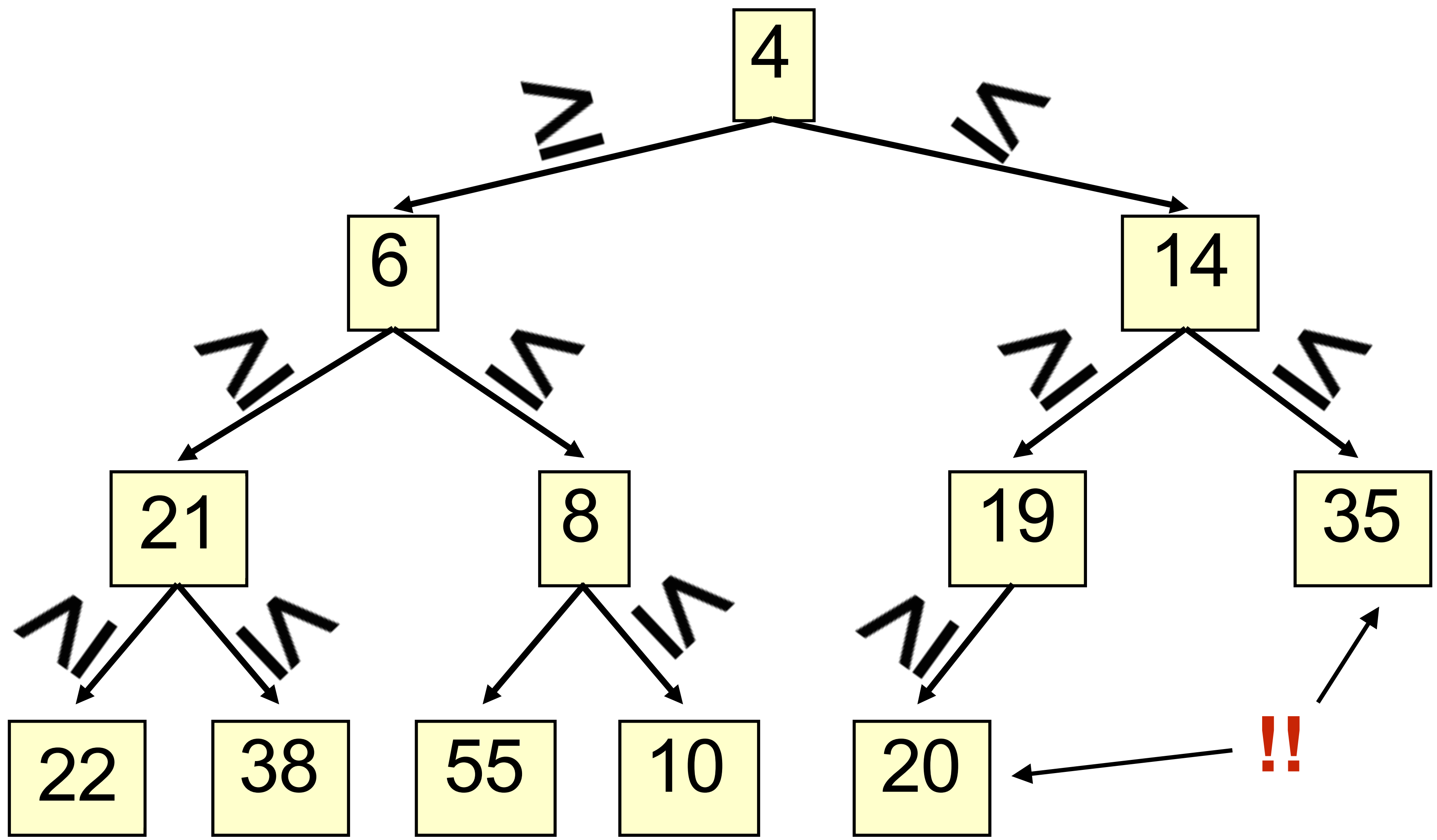
Priority Queue: heap implementation

- A heap is a **concrete** data structure that can be used to **implement** a Priority Queue
- Better runtime complexity than either list implementation:
  - **peek()** is O(1)
  - **poll()** is O(log n)
  - **add()** is O(log n)
- Not to be confused with *heap memory*, where the Java virtual machine allocates space for objects – different usage of the word heap.

A heap is a special binary tree with two additional properties.

A heap is a special binary tree.

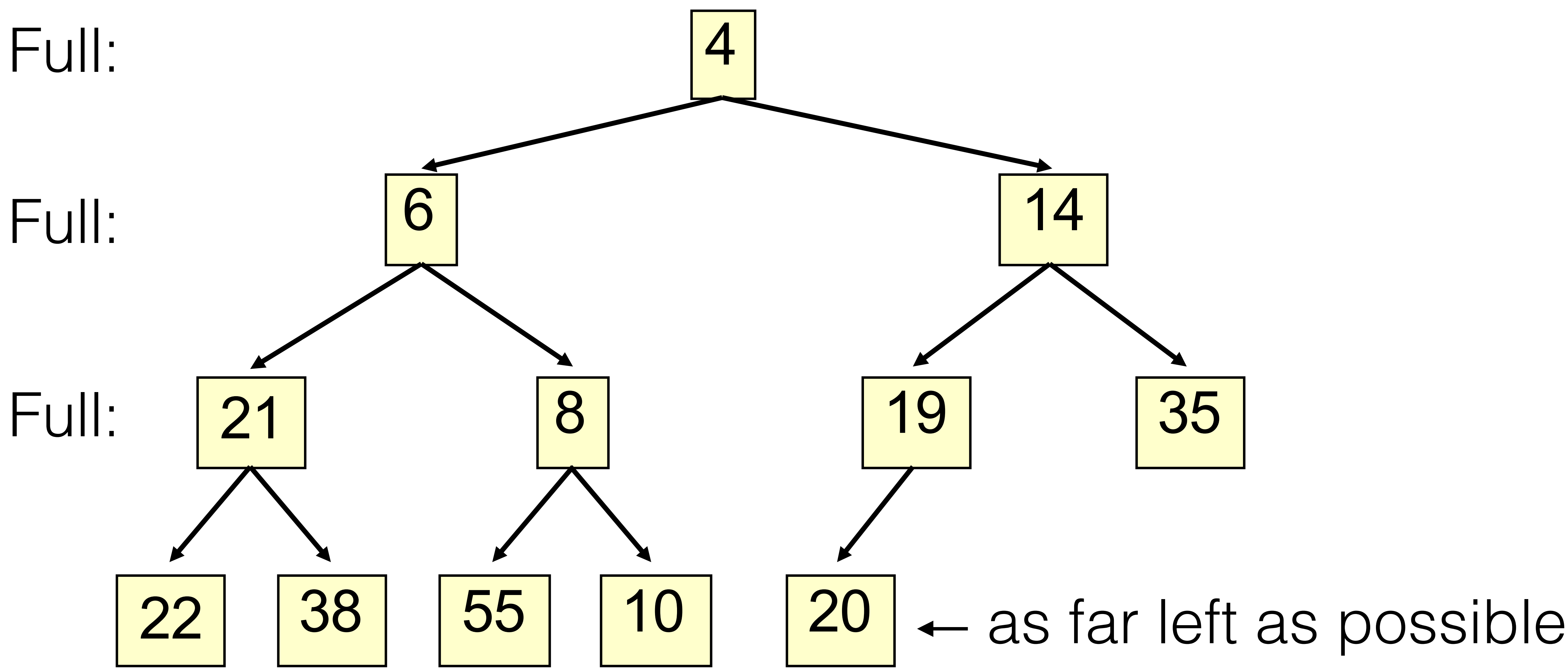
1. **Heap Order Invariant:**
Each element $\geq$ its parent.



A heap is a special binary tree.

2. **Complete:** no holes!

- All levels except the last are full.
- Nodes in last level are as far left as possible.



Heap it real

```
    (A)
     5
    /   \
  12      15
 / \
13 11
```

```
    (B)
     5
    /
  12
 / \
13 14
```

```
      (C)
       5
      /   \
    12      15
   /       / \
  13      15  16
```

```
      (D)
      -5
     /    \
   12      15
  / \      /
18 13    15
```

Which of these are valid heaps?

Heap it real.

```
    (A)
     5
   /   \
 12      15
 / \
13 11
```

```
   (B)
     5
    /
  12
  / \
13 14
```

```
     (C)
      5
    /   \
  12     15
  /     /  \
13    15    16
```

```
    (D)
     -5
   /     \
 12       15
 / \      /
18 13   15
```

Which of these are valid heaps?

Heap operations

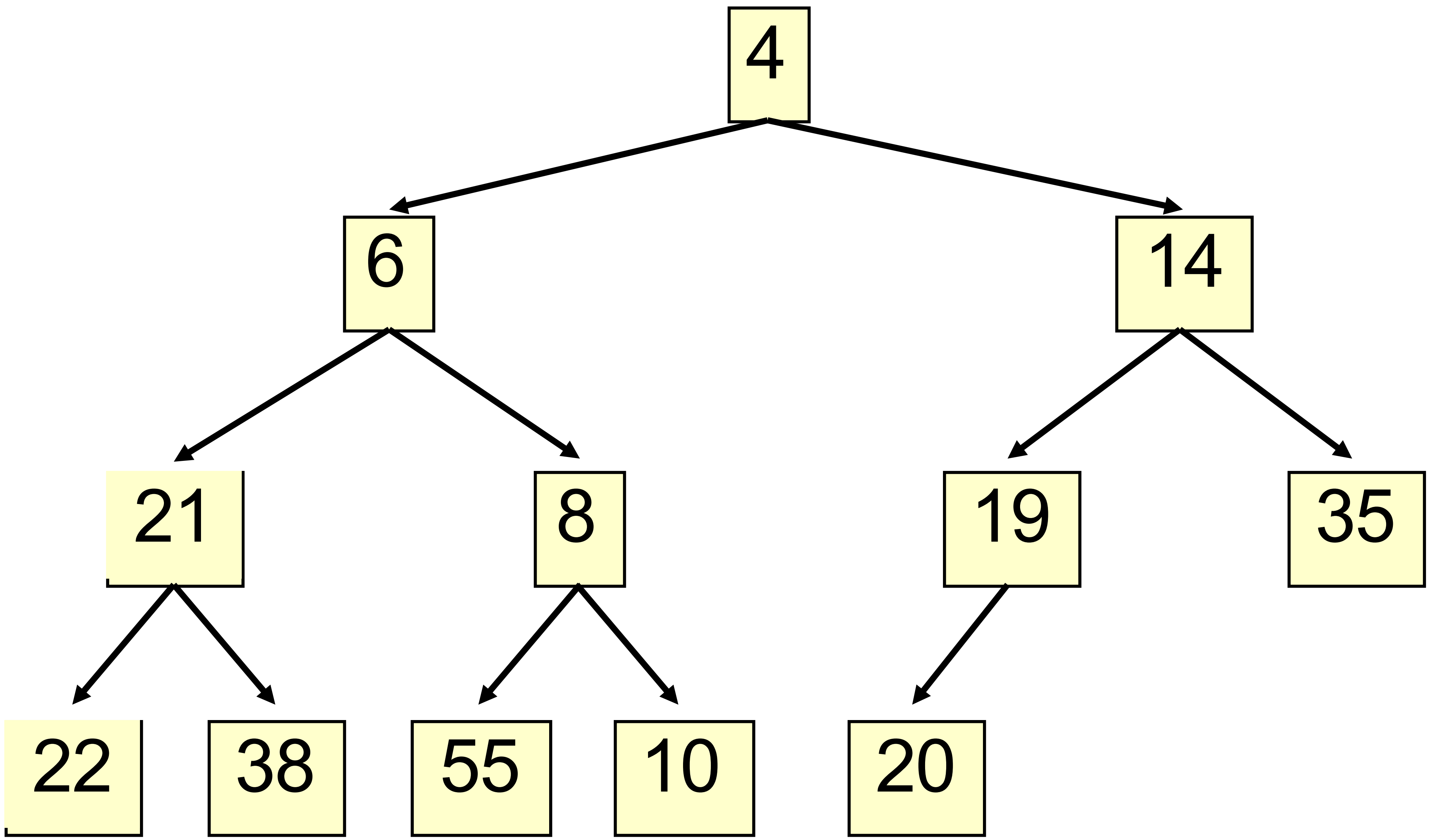
```
interface PriorityQueue<E> {
 boolean add(E e); // insert e
 E peek(); // return min element
 E poll(); // remove/return min element
 void clear();
 boolean contains(E e);
 boolean remove(E e);
 int size();
 Iterator<E> iterator();
}
```

```java
boolean add(E e);
```

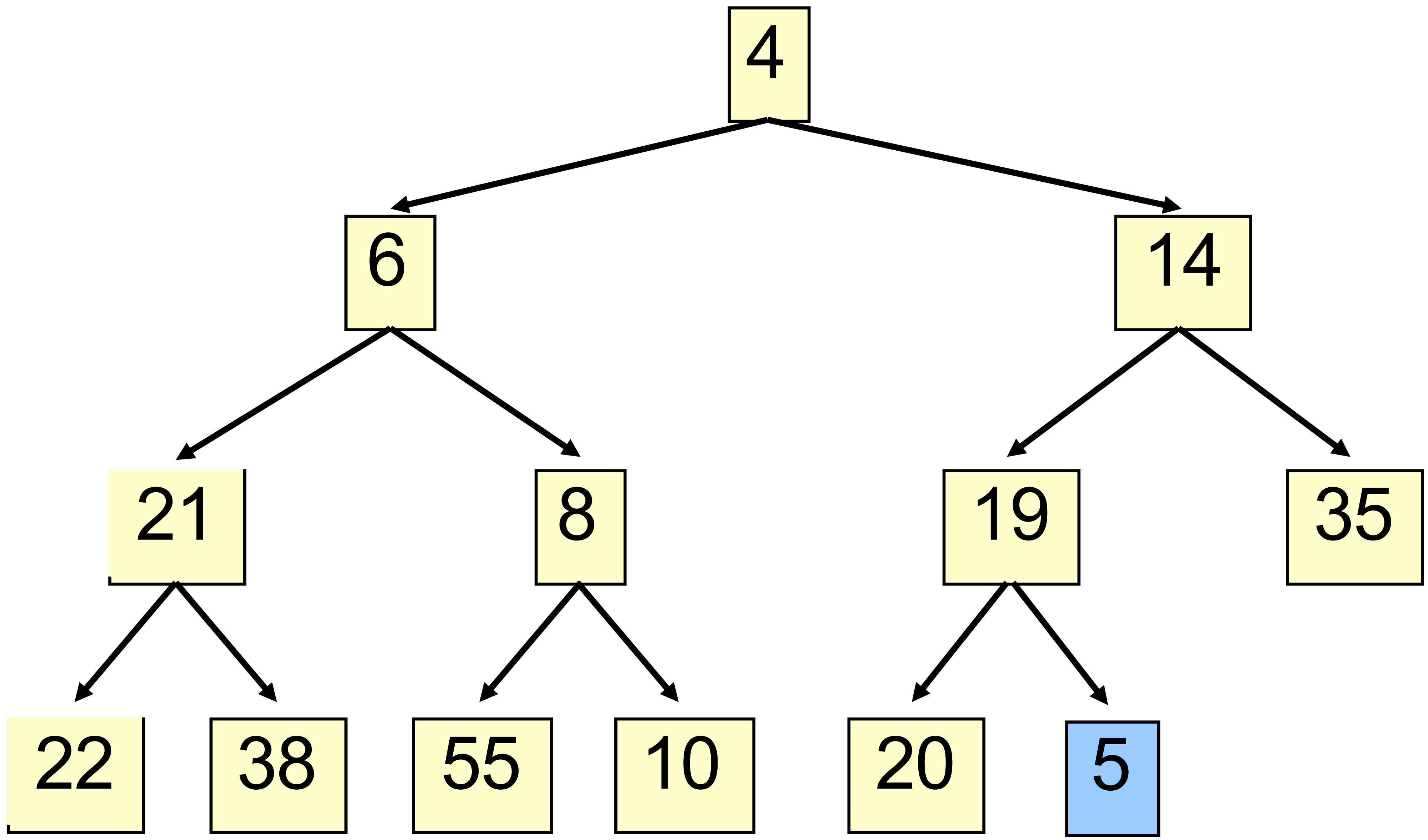
Algorithm:

- Add e in the wrong place
- While e is in the wrong place
  - move e towards the right place

```
boolean add(E e);
```



`boolean add(E e);`



```
boolean add(E e);
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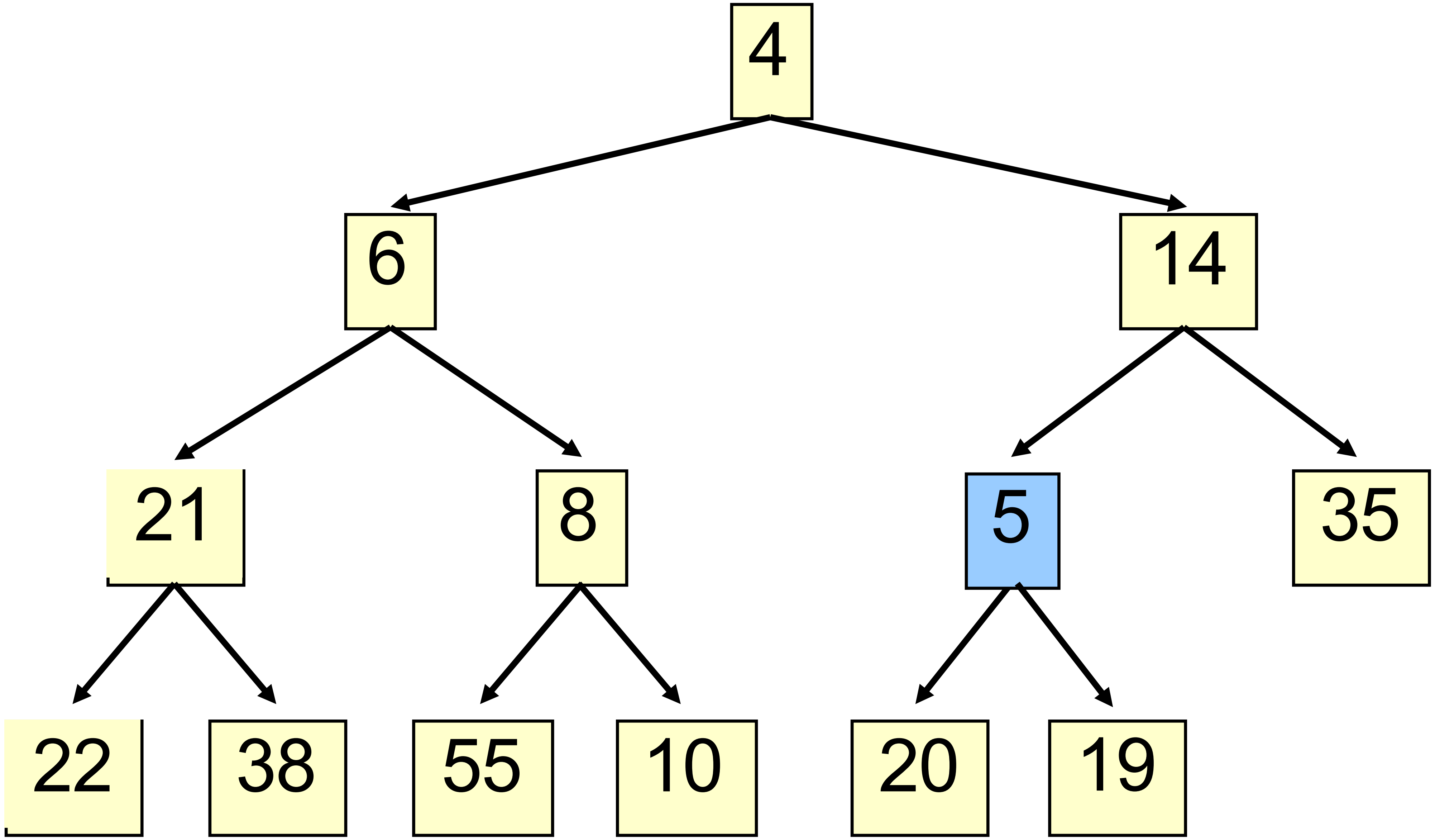

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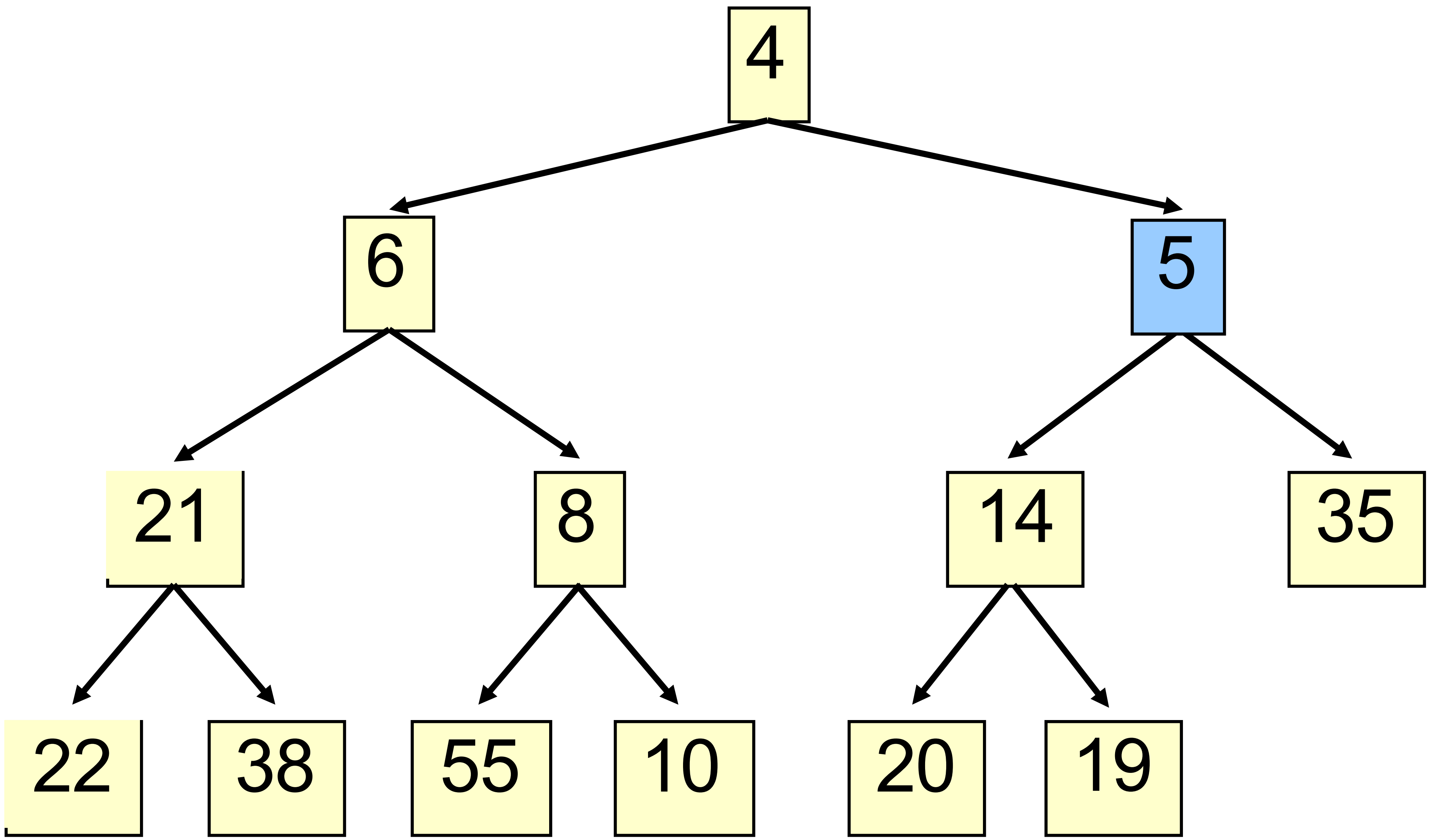

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boolean add(E e);
```
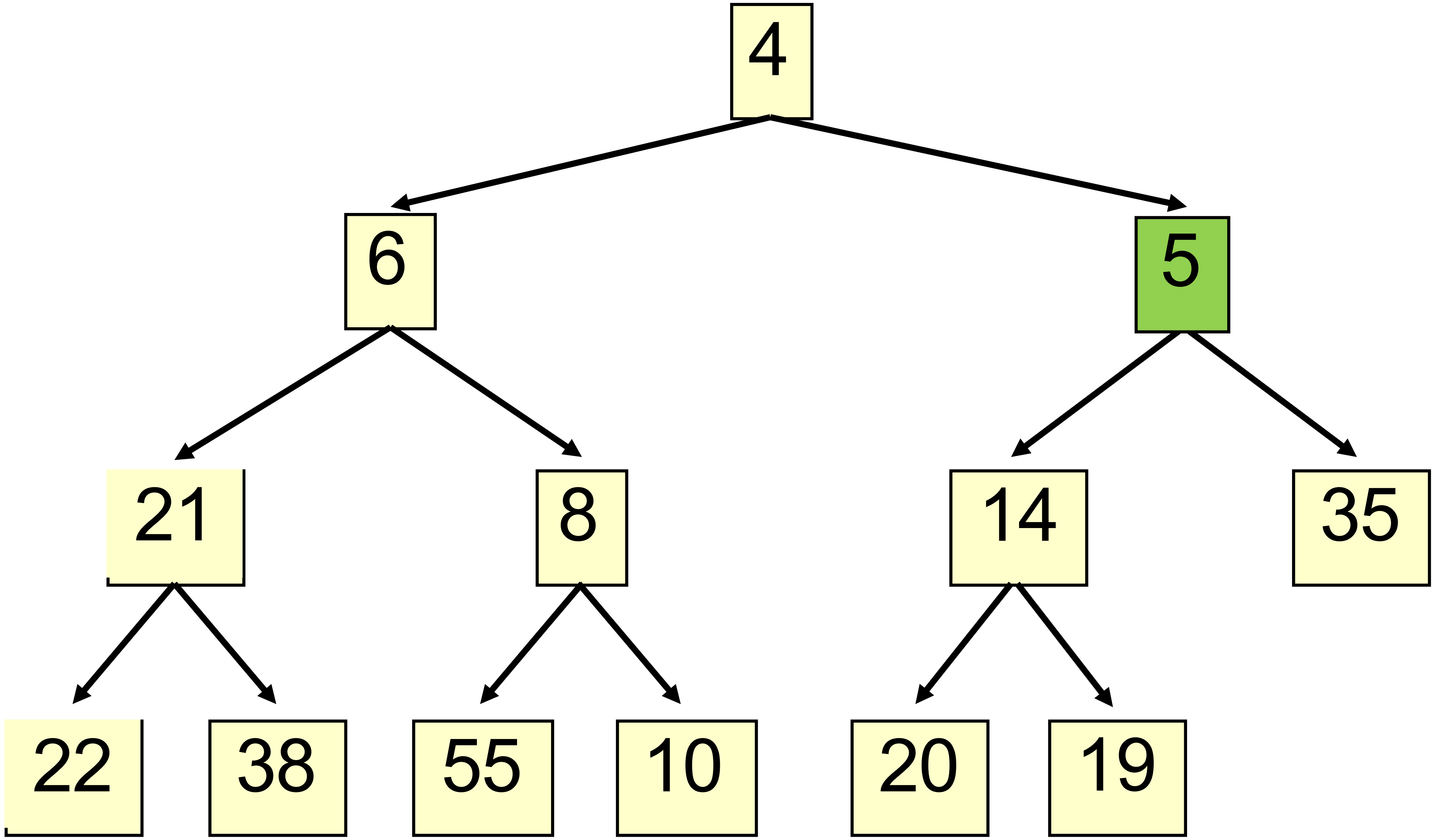



```
boolean add(E e);
```

Algorithm:

- Add e in the wrong place (the leftmost empty leaf)
- While e is in the wrong place (it is less than its parent)
  - move e towards the right place (swap with parent)

The heap invariant is maintained!

What's the runtime?

- O(number of swap/bubble operations)

= O(height of tree)

- Complete => balanced

  =>height is **O(log n)**

- Maximum number of swaps is O(log n)

add(e)

Algorithm:

- Add e in the wrong place (the leftmost empty leaf)
- While e is in the wrong place (it is less than its parent)
  - move e towards the right place (swap with parent)

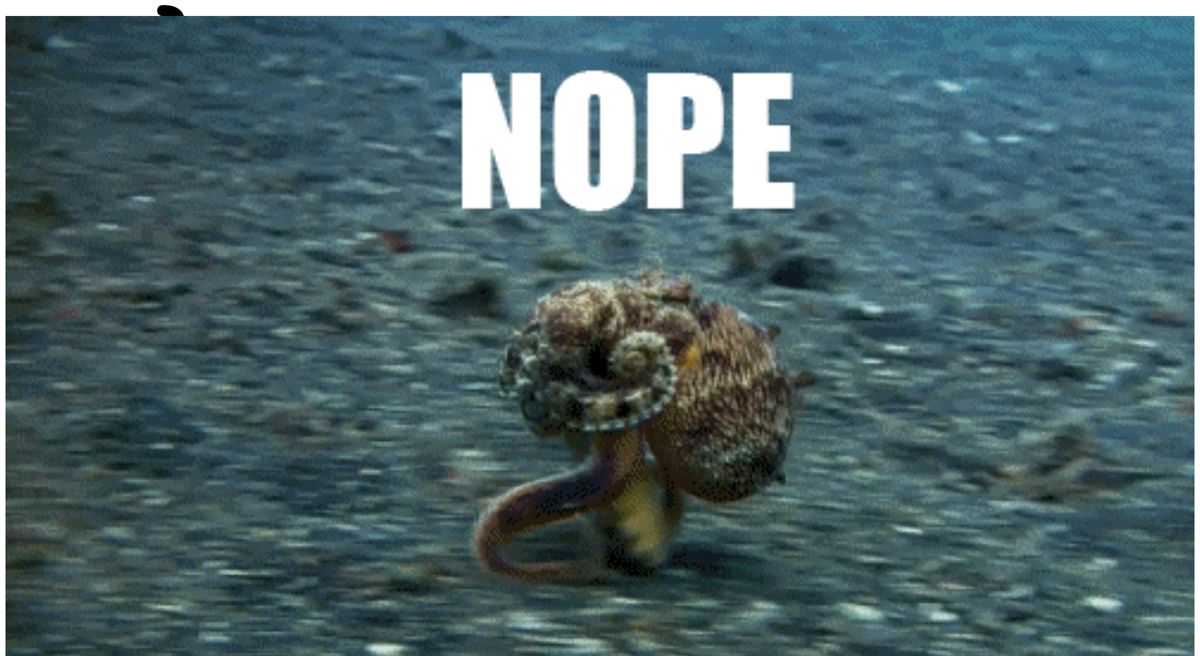
The heap invariant is maintained!

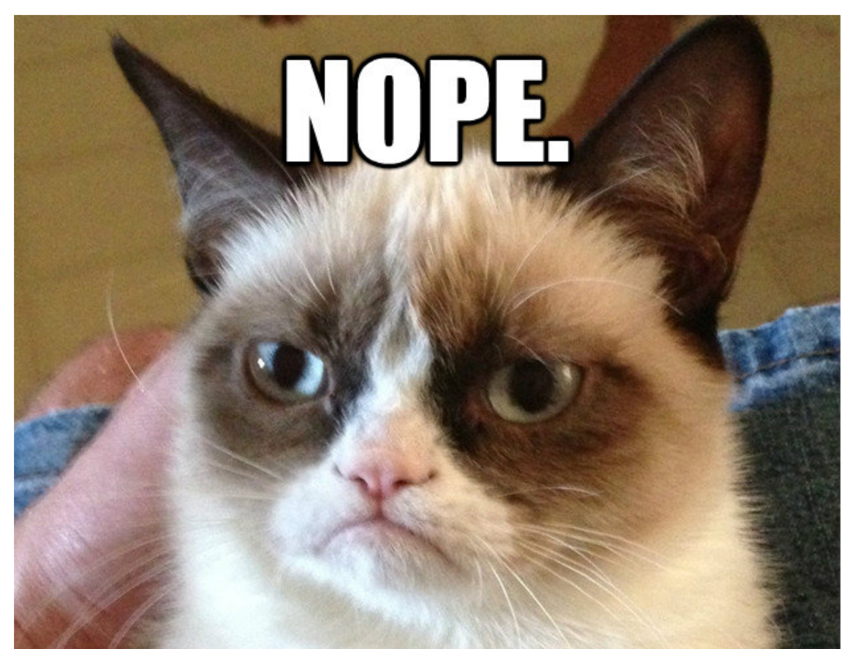
Implementing Heaps

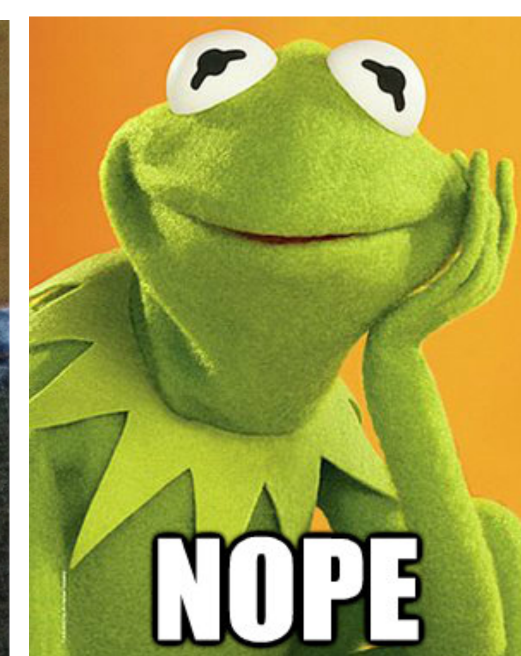
```
public class HeapNode {
   private int value;
   private HeapNode left;
   private HeapNode right;
   ...
}
public class Heap {
   HeapNode root;
   ...
```

Implementing Heaps

```
public class HeapNope {
   private int value;
   private HeapNope left;
   private HeapNope right;
   ...
}
```

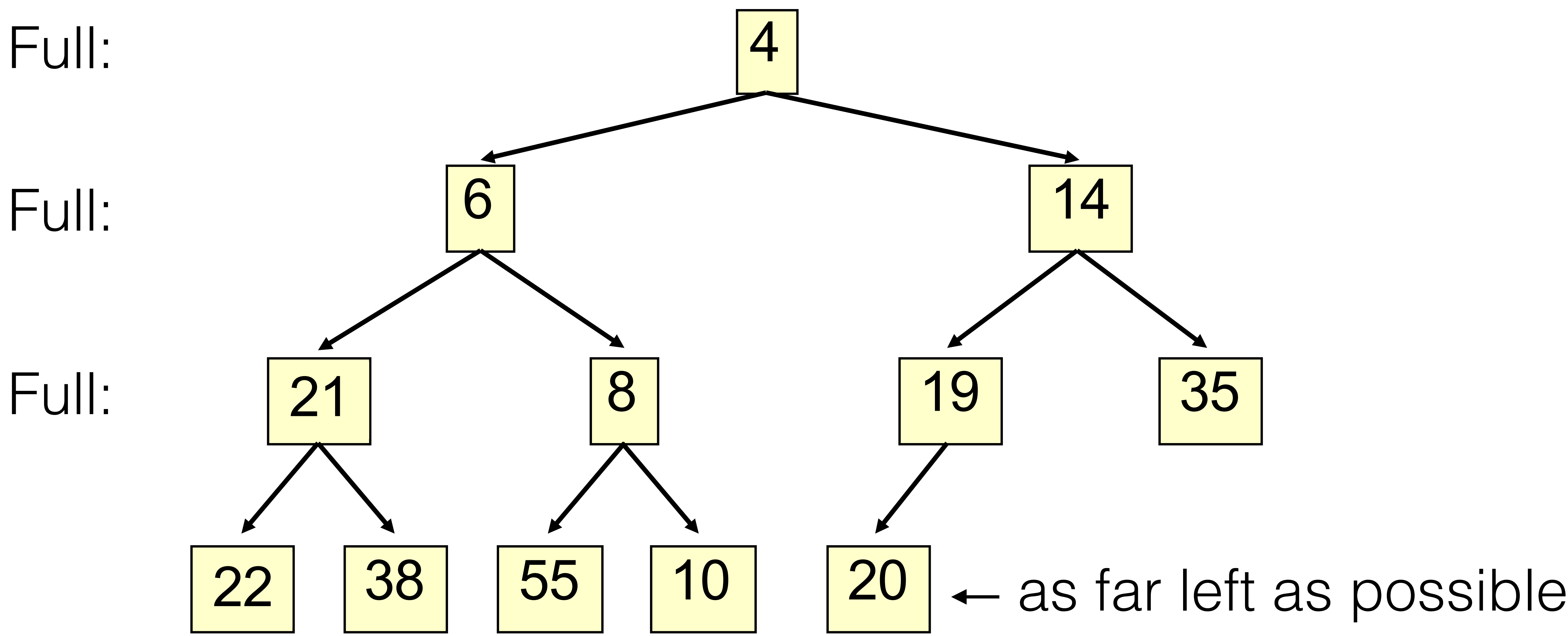






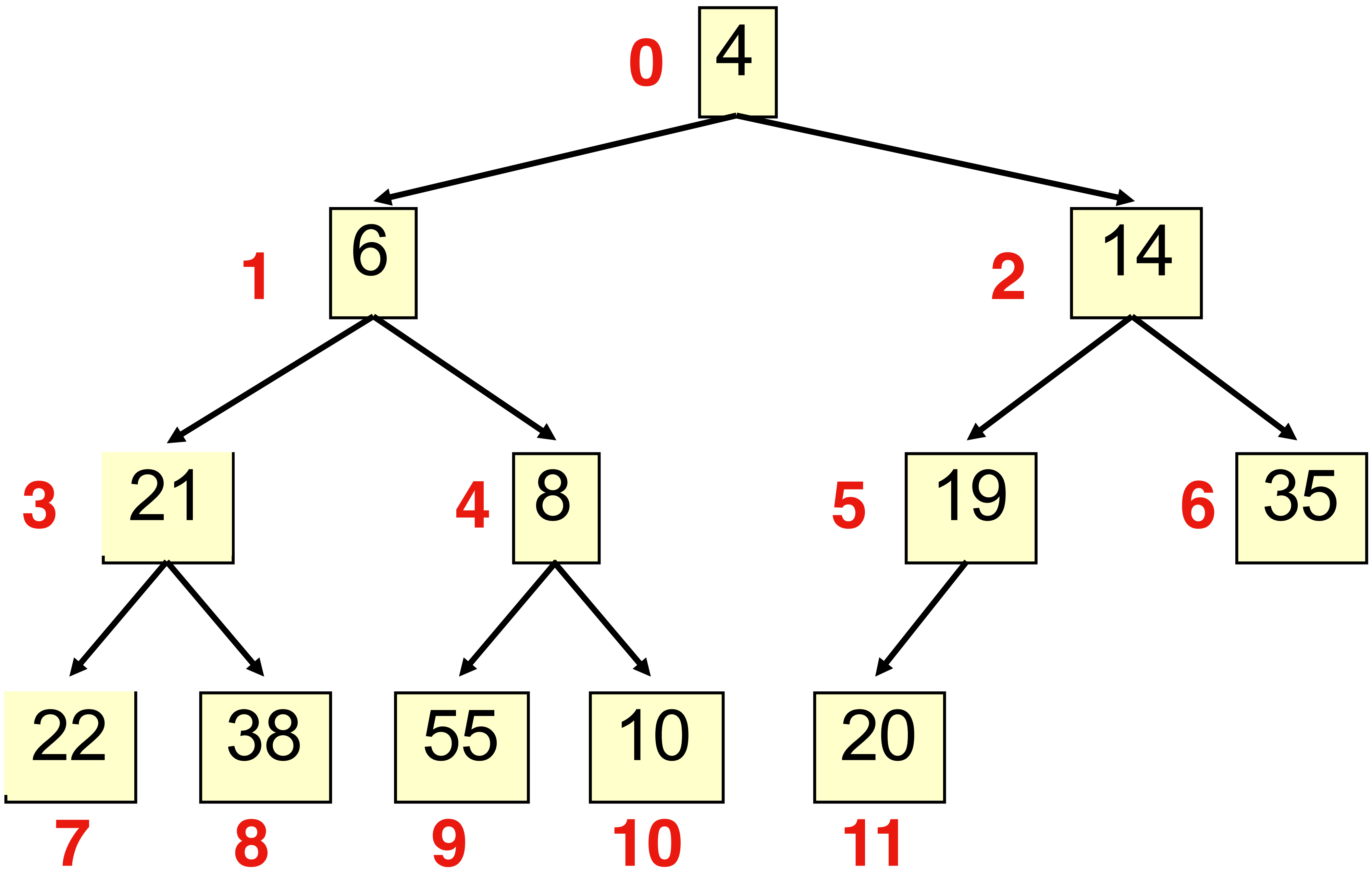
A heap is a special binary tree.

2. **Complete:** no holes!



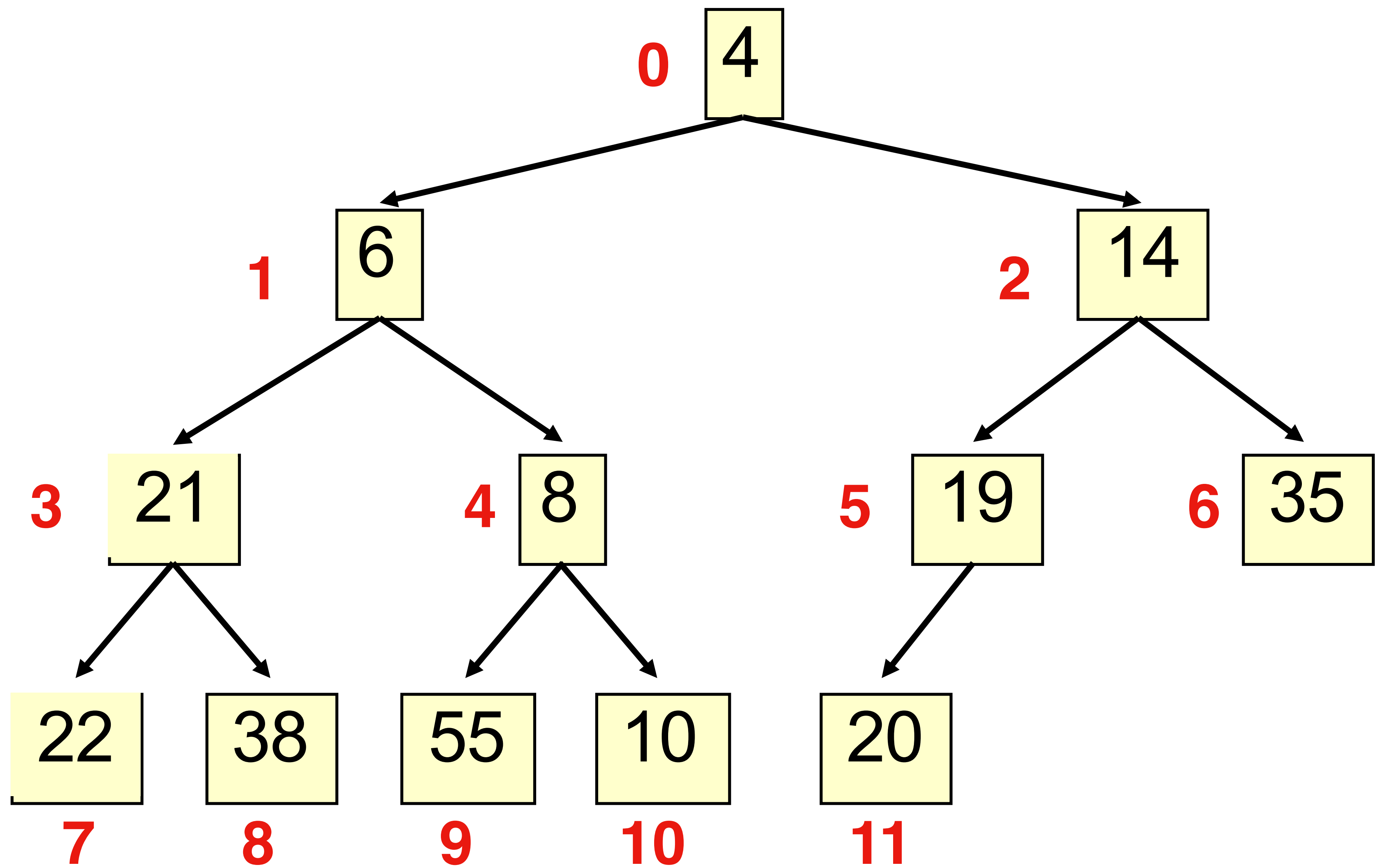
Numbering Nodes

Level-order traversal:



2. Complete: **no holes!**

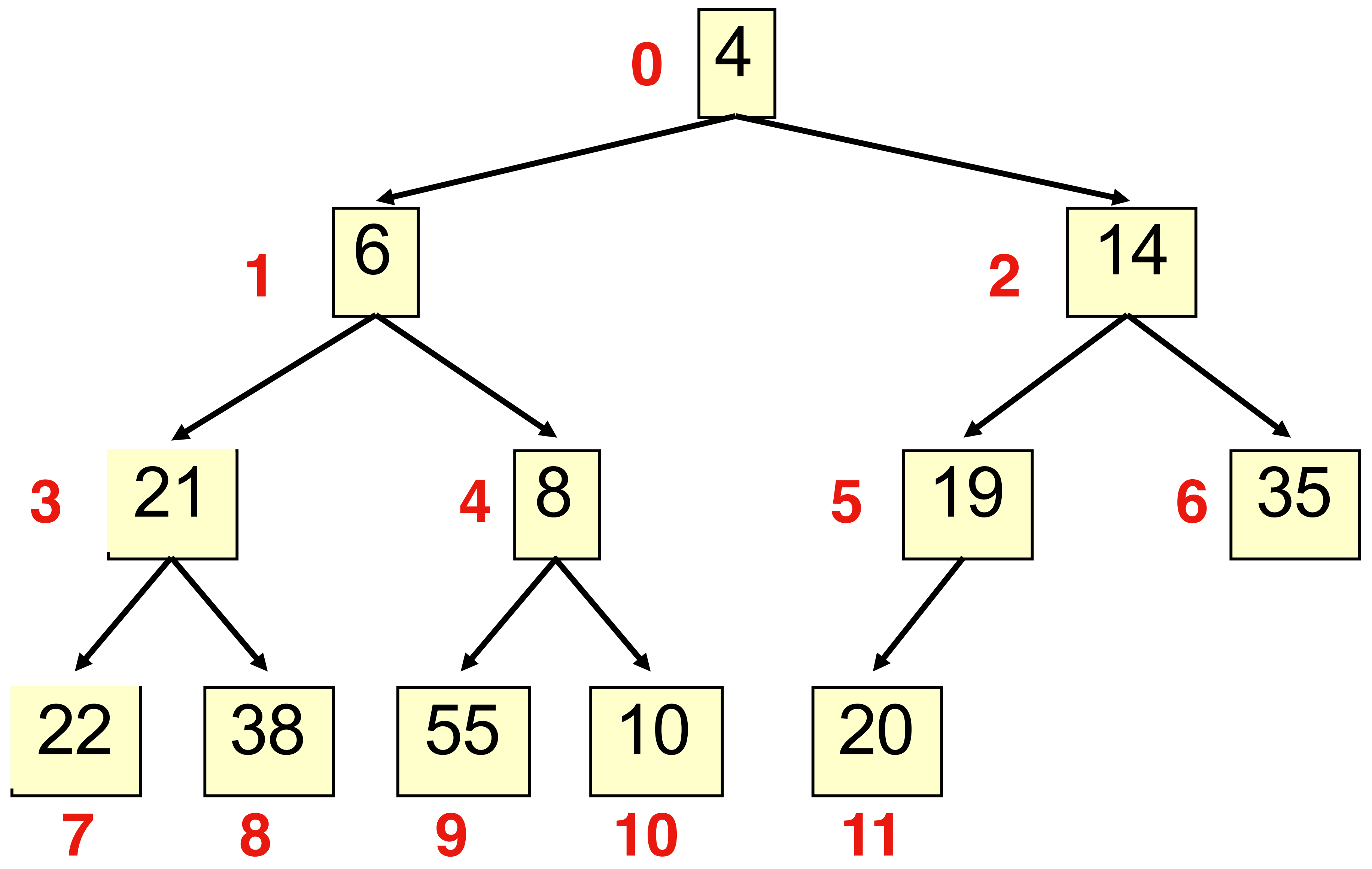
Numbering Nodes



node **k**’s parent is

node **k**’s children are nodes and

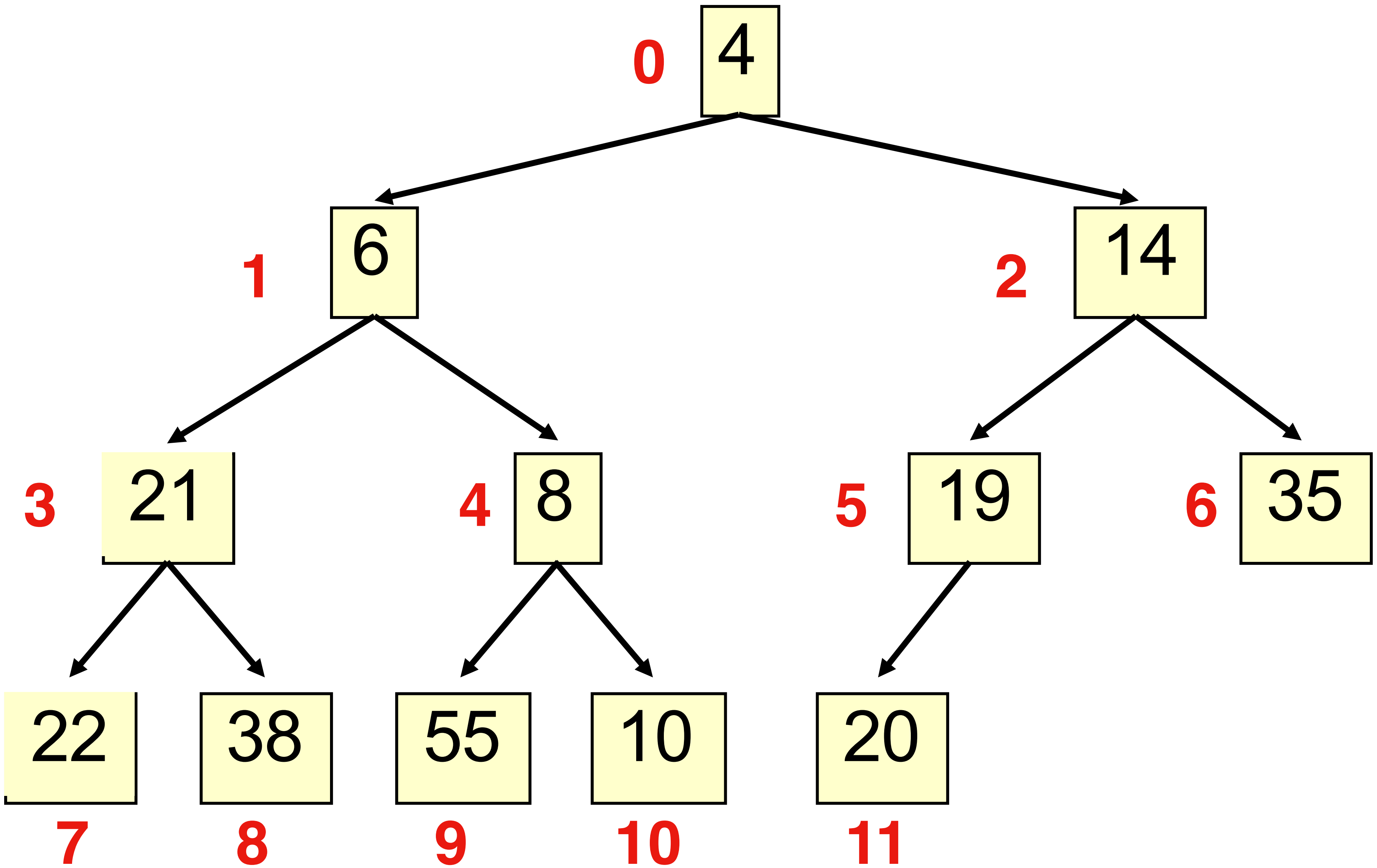
Numbering Nodes



node **k**’s parent is **(k – 1)/2**

node **k**’s children are nodes and

Numbering Nodes



node **k**'s parent is **(k – 1)/2**

node **k**'s children are nodes **2k + 1** and **2k + 2**

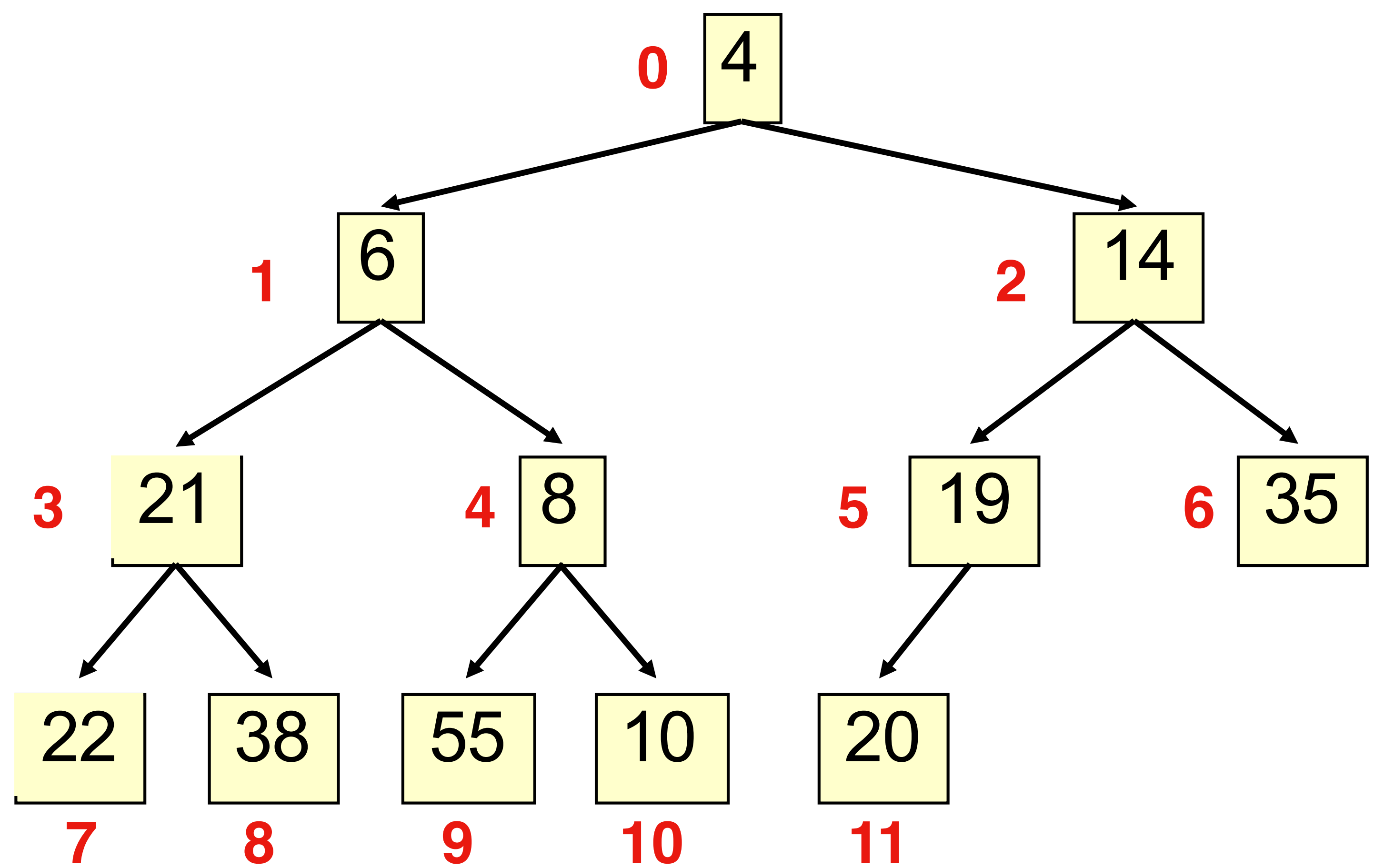
Implementing Heaps

```
public class Heap<E> {
  private E[] heap;
  private int size;
  ...
}
```

0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
4	6	14	21	8	19	35	22	38	55	10	20				

Implicit Tree Structure

2. Complete: **no holes!**



0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
4	6	14	21	8	19	35	22	38	55	10	20				