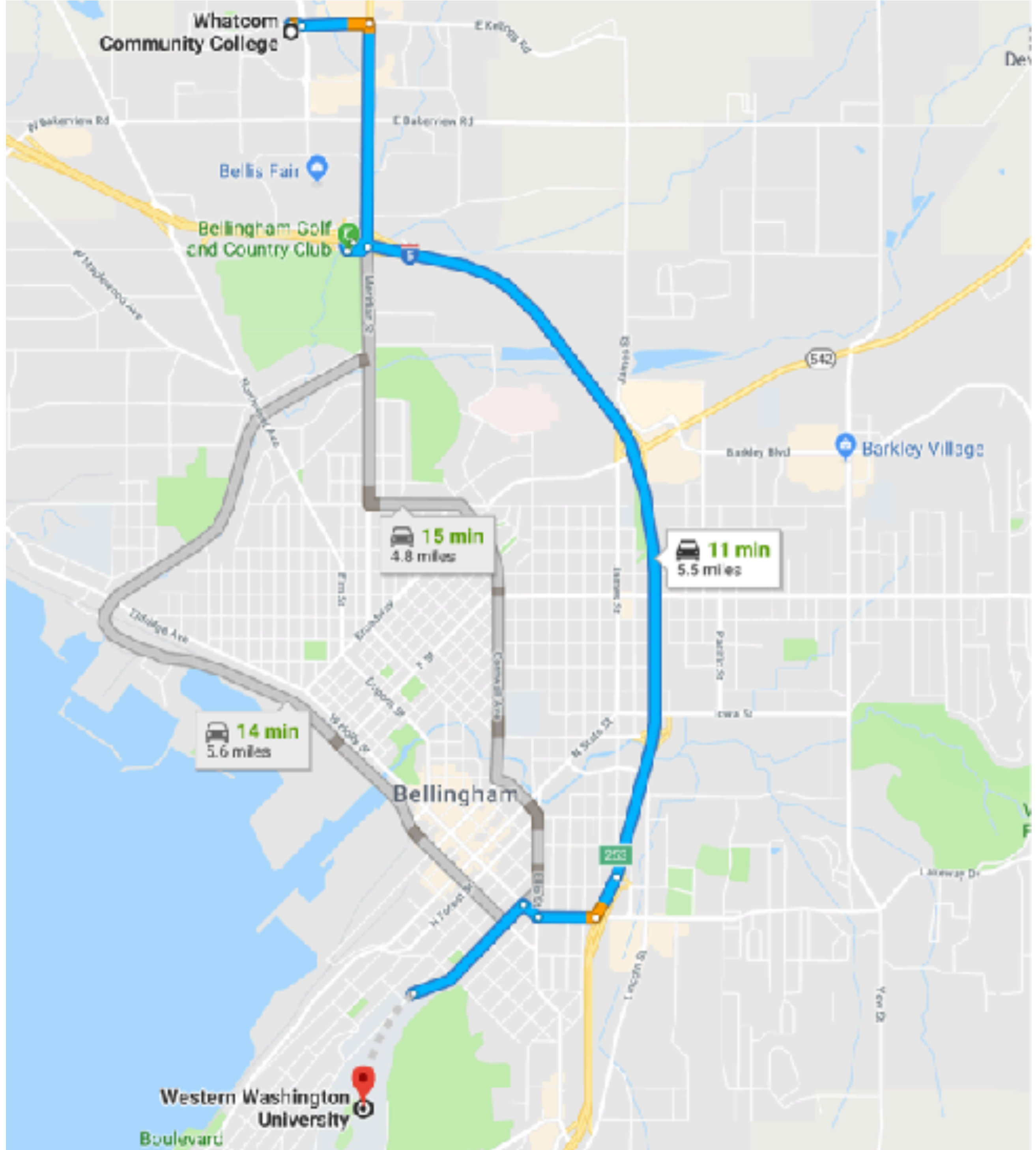


CSCI 241

Lecture 22

Dijkstra's Single-Source Shortest Paths Algorithm

Announcements

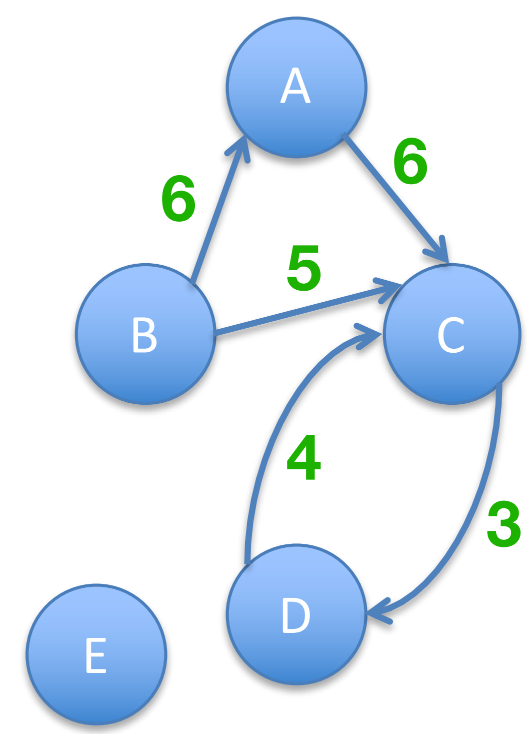
- Nick has office hours today 2-4 CF 163.
- hours.txt should contain a single integer. No more, no less.
- Nontrivial code sections (such as rebalance and rotations) should have comments!
- Test that your code compiles and runs on the command line:
  - `git clone your_repo_url && cd your_repo_name`
  - `make test2`
  - `make test1 # yes, you have to run test2 before test1`
  - `make test3`

Goals

- Know what a weighted graph is.
- Understand the intuition behind Dijkstra’s shortest paths algorithm.
- Be able to execute Dijkstra’s algorithm manually on a graph.

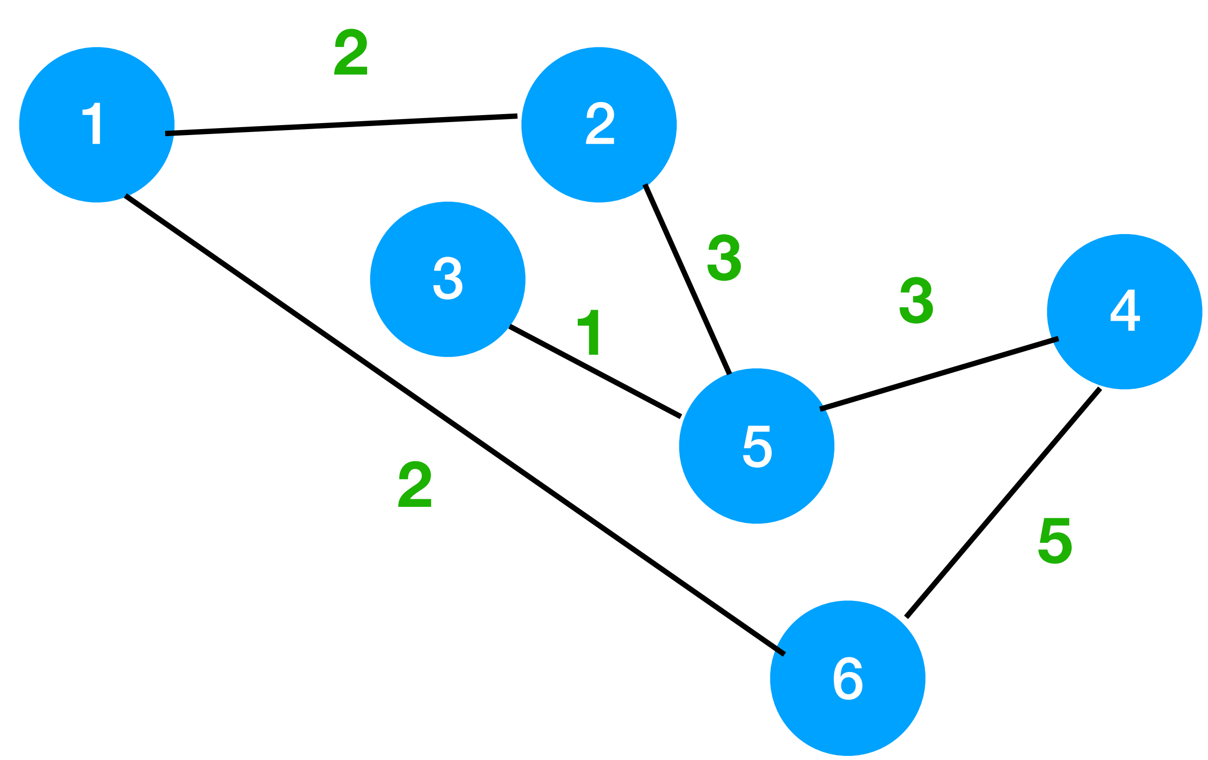
Weighted Graphs

- Like a normal graph, but edges have weights.
- Formally: a graph (V,E) with an accompanying weight function w: E -> $\mathbb{R}$
  - may be directed or undirected.
- Informally: label edges with their weights
- Representation:
  - adjacency list - store weight of (u,v) with v the node in u's list
  - adjacency matrix - store weight in matrix entry for (u,v)



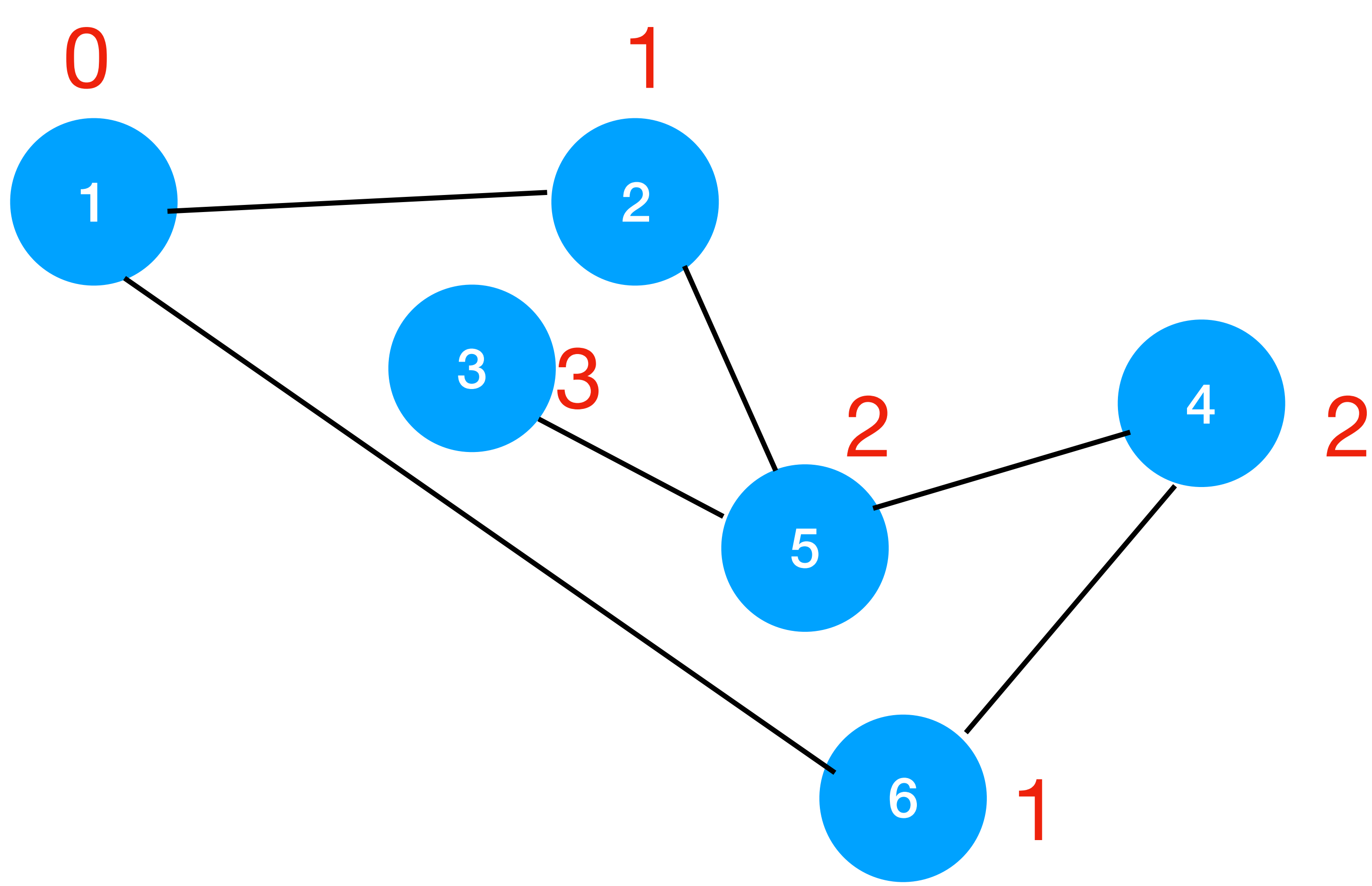
Paths in Weighted Graphs

- The length (or weight) of a path in a weighted graph is the sum of the edge weights along that path.
- **ABCD**: What's the length of the shortest path from 3 to 6?
  A. 7
  B. 8
  C. 9
  D. 10



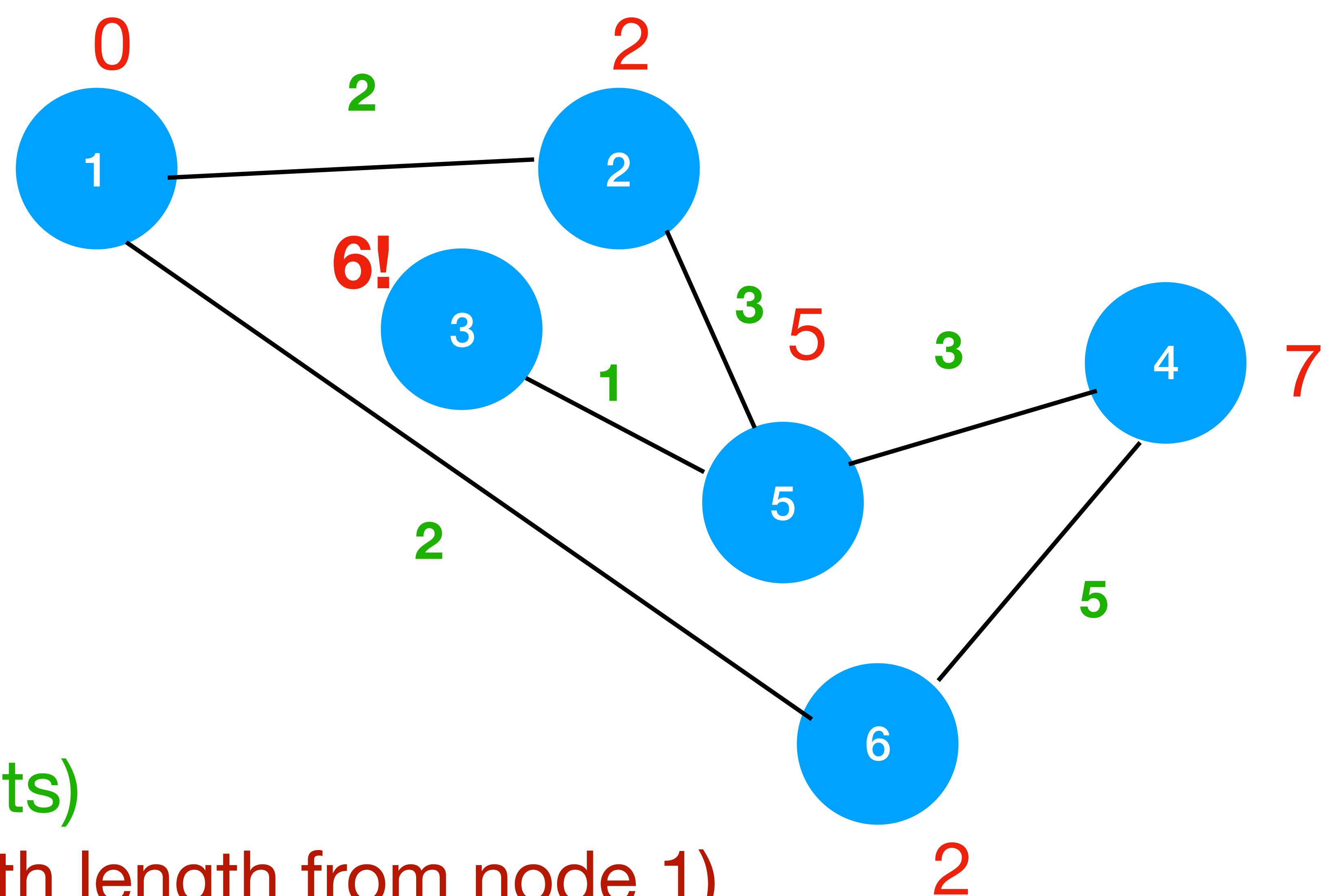
Computing Shortest Paths in Unweighted Graphs

- Perform a breadth-first search (that's it!)
- BFS visits nodes in order of "hop distance", or path length!
- BFS(1):



Computing Shortest Paths in Weighted Graphs

BFS doesn't visit nodes in order of shortest path length:



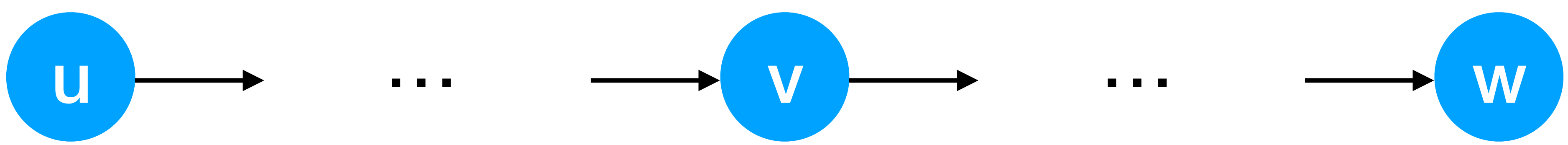
(edge weights)
(shortest path length from node 1)

Dijkstra’s Shortest Paths: History

- When Dijkstra designed the algorithm in 1956 (at age 26), most people were programming in assembly language.
- Fortran was the only high-level language, and it wasn’t quite finished at the time.
- Big-O analysis had not been thought of yet. In his paper, Dijkstra says, “my solution is preferred to another one … “the amount of work to be done seems considerably less.”

Dijkstra’s Shortest Paths: Subpaths

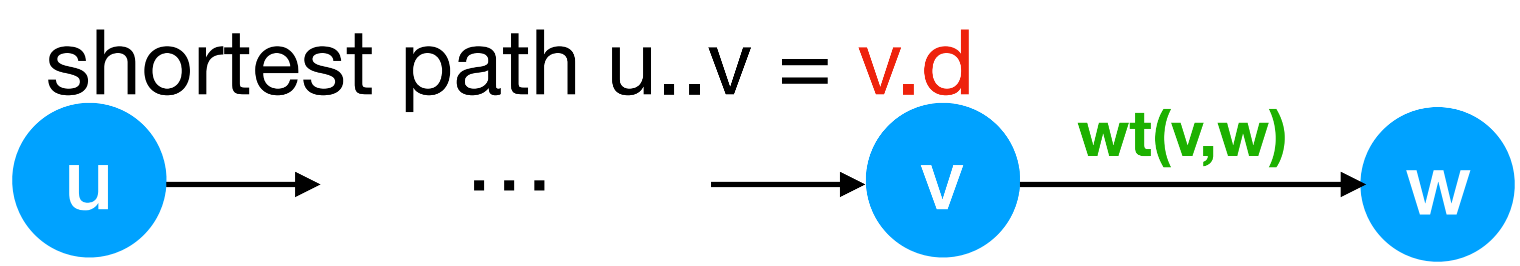
- Fact: **subpaths** of shortest paths are shortest paths



- Example: if the shortest path from u to w goes through v, then
  - the part of that path from u to v is the shortest path from u to v.
  - if there were some better path u..v, that would also be part of a better way to get from u to w.

Dijkstra's Shortest Paths: Subpaths

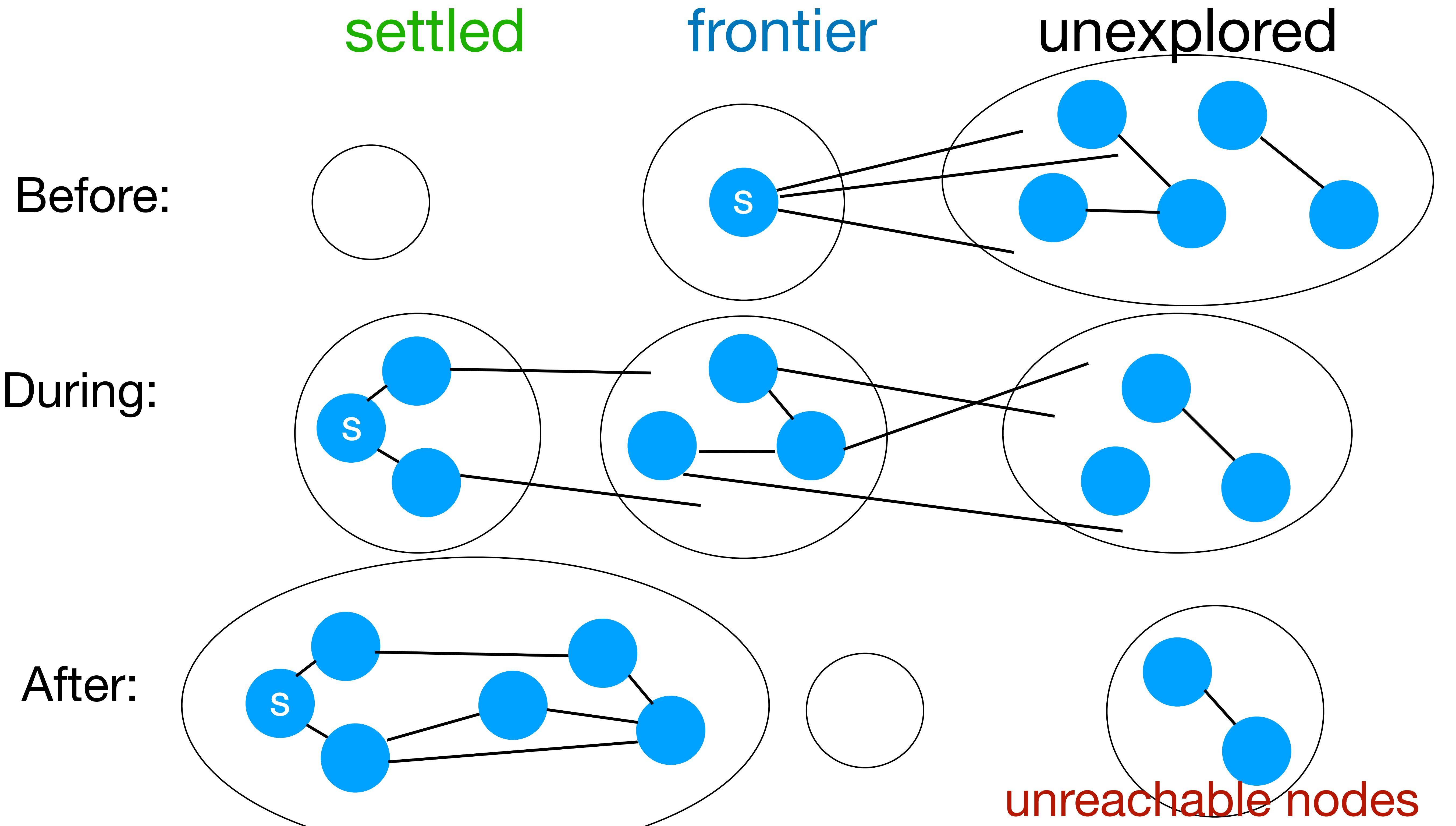
- Fact: **subpaths** of shortest paths are shortest paths
- Consequence: a **candidate** shortest path from start node **s** to some node **v**'s neighbor **w** is the shortest path from to **v** + the edge weight from **v** to **w**.



Dijkstra's Shortest Paths: Intuition

- Intuition: explore nodes kinda like BFS.
- There are three kinds of nodes:
  - Settled - nodes for which we know the actual shortest path.
  - Frontier - nodes that have been visited but we don't necessarily have their actual shortest path
  - Unexplored - all other nodes.
- Each node n keeps track of `n.d`, the length of the shortest known known path from start.
- We may discover a shorter path to a frontier node than the one we've found already - if so, update `n.d`.

Dijkstra's Shortest Paths: Cartoon



Dijkstra's Shortest Paths: High-Level Algorithm

```
Initialize Settled to empty
Initialize Frontier to the start node
While the frontier isn't empty:
  move the node f with smallest d from F to S
  For each neighbor w of f:
    if we've never seen w before:
      set its path length
      add it to frontier
  else if the path to w via f is shorter:
      update w's shortest path length
```

Dijkstra's Shortest Paths: High-Level Algorithm

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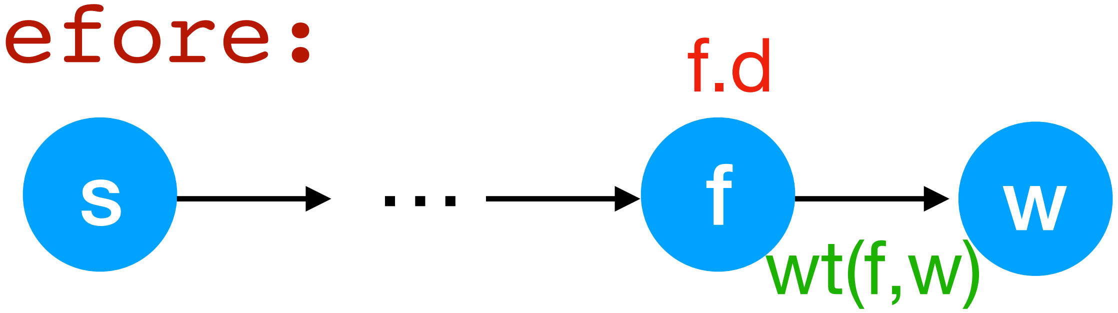

Dijkstra's Shortest Paths: High-Level Algorithm

```
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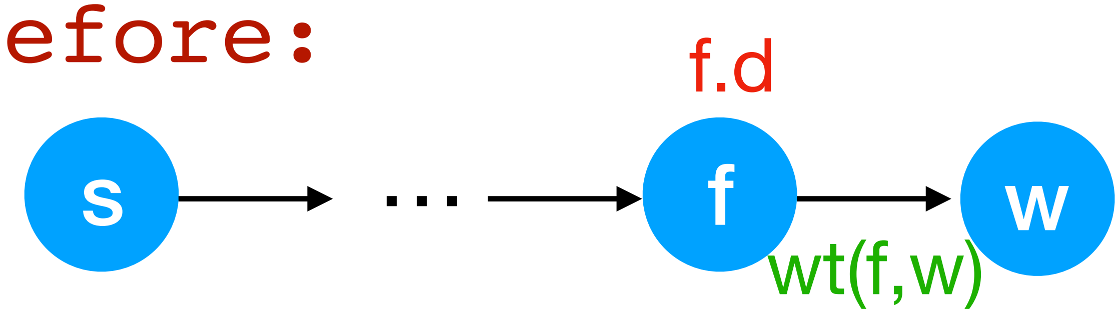
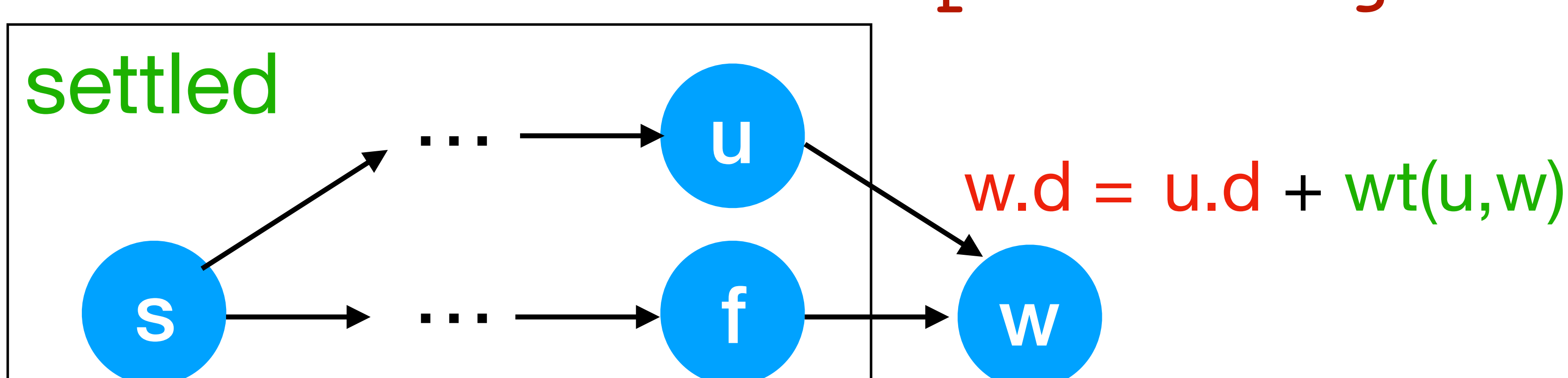
```





Dijkstra’s Shortest Paths: High-Level Algorithm

```
Initialize Settled to empty
Initialize Frontier to the start node
While the frontier isn’t empty:
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      if we’ve never seen w before:
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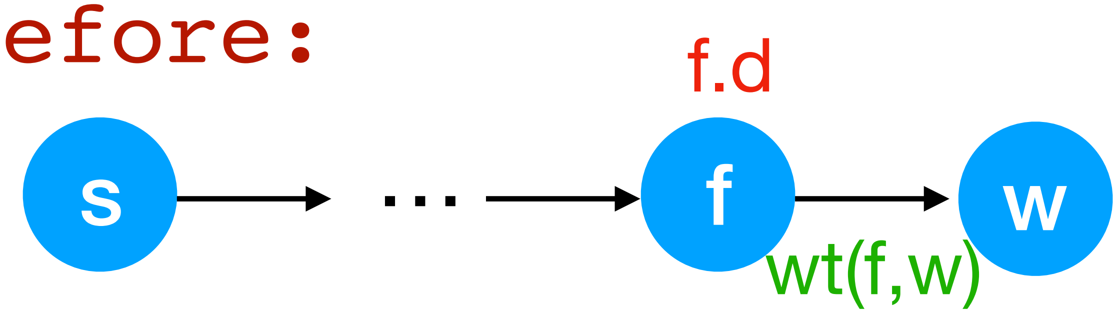
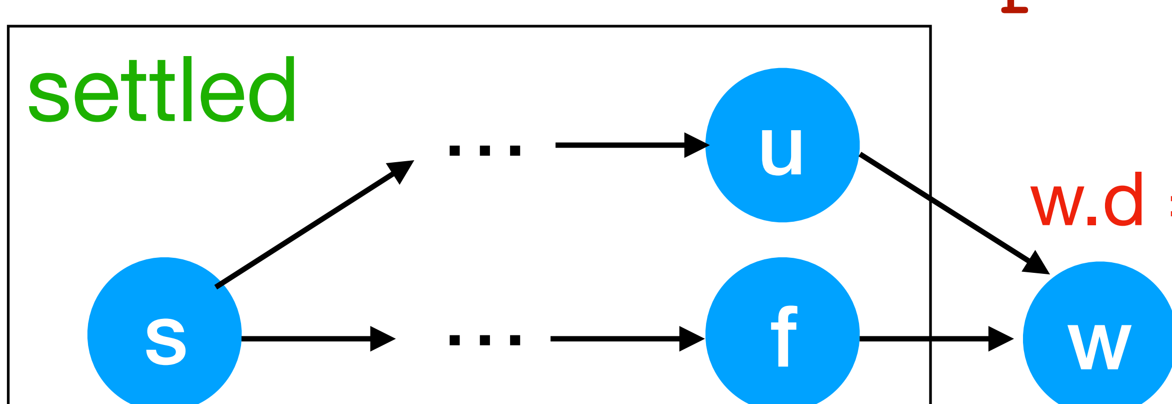
         add it to frontier
   else if the path to w via f is shorter:
         update w’s shortest path length
```





w.d = ~~u.d + wt(u,w)~~

f.d + wt(f,w)

Dijkstra’s Shortest Paths: Execution

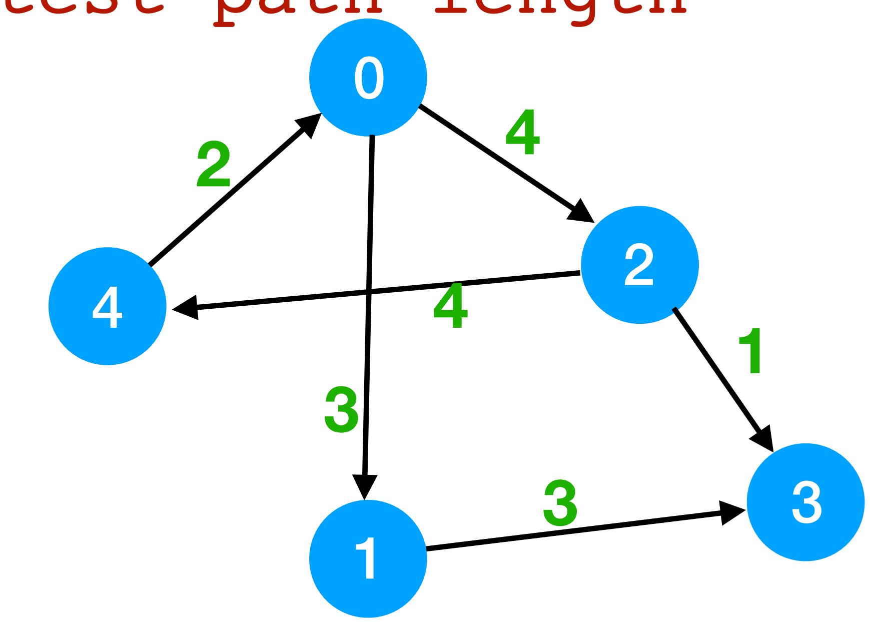
Best known distances:

Node	d
0	?
1	?
2	?
3	?
4	?

```
Initialize Settled to empty
Initialize Frontier to the start node
While the frontier isn't empty:
  move the node f with smallest d from F to S
  For each neighbor w of f:
    if we've never seen w before:
      set its path length to f.d + wt(f,w)
      add w to the frontier
    else if the path to w via f is shorter:
      update w's shortest path length
```

Settled set:

Frontier set:



shortest-paths(4)

Dijkstra's Shortest Paths: Execution

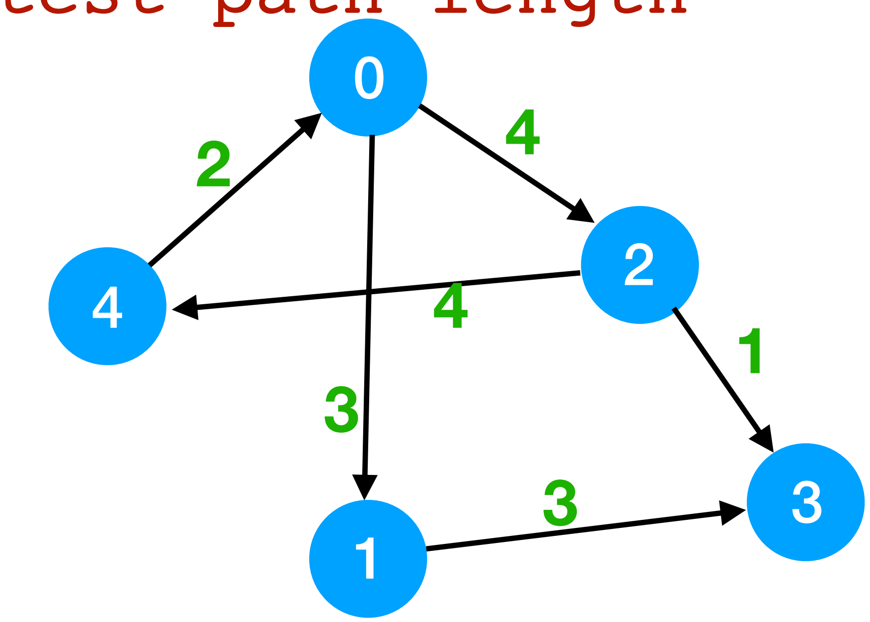
Best known distances:

Node	d
0	?
1	?
2	?
3	?
4	0

```
Initialize Settled to empty
Initialize Frontier to the start node
While the frontier isn't empty:
   move the node f with smallest d from F to S
   For each neighbor w of f:
      if we've never seen w before:
         set its path length to f.d + wt(f,w)
         add w to the frontier
      else if the path to w via f is shorter:
         update w's shortest path length
```

Settled set: {}

Frontier set: {4}



shortest-paths(4)

Dijkstra's Shortest Paths: Execution

Best known distances:

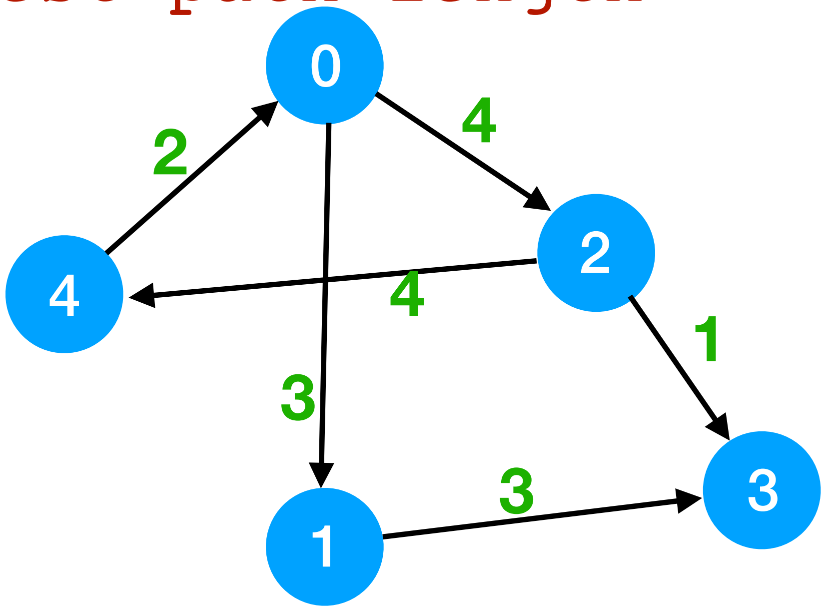
Node	d
0	?
1	?
2	?
3	?
4	0

```
Initialize Settled to empty
Initialize Frontier to the start node
While the frontier isn't empty:
  move the node f with smallest d from F to S
  For each neighbor w of f:
    if we've never seen w before:
      set its path length to f.d + wt(f,w)
      add w to the frontier
    else if the path to w via f is shorter:
      update w's shortest path length
```

f: 4

Settled set: {4}

Frontier set: {}



shortest-paths(4)

Dijkstra's Shortest Paths: Execution

Best known distances:

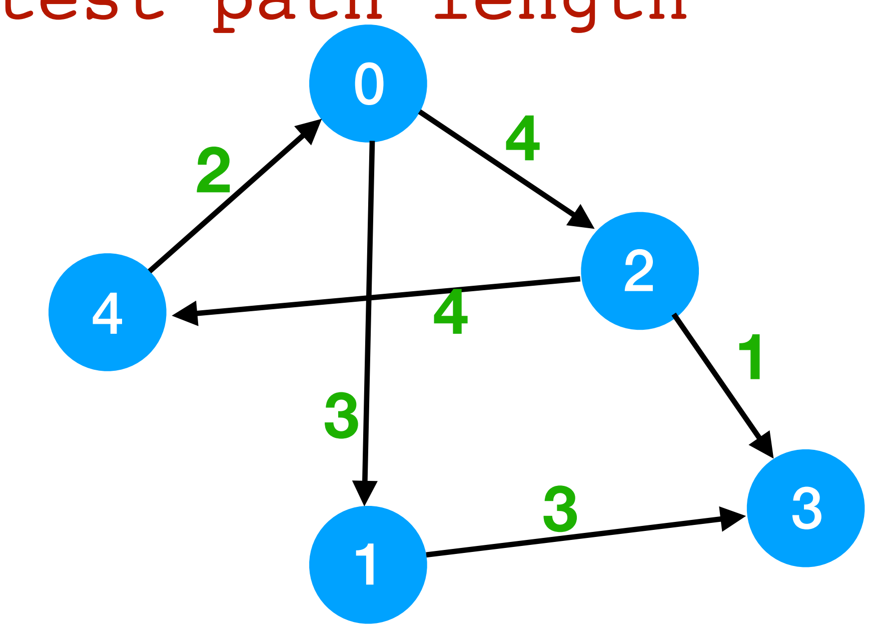
Node	d
0	2
1	?
2	?
3	?
4	0

```
Initialize Settled to empty
Initialize Frontier to the start node
While the frontier isn't empty:
   move the node f with smallest d from F to S
   For each neighbor w of f:
     if we've never seen w before:
       set its path length to f.d + wt(f,w)
       add w to the frontier
     else if the path to w via f is shorter:
       update w's shortest path length
```

f: 4
w: 0

Settled set: {4}

Frontier set: {0}



shortest-paths(4)

Dijkstra's Shortest Paths: Execution

Best known distances:

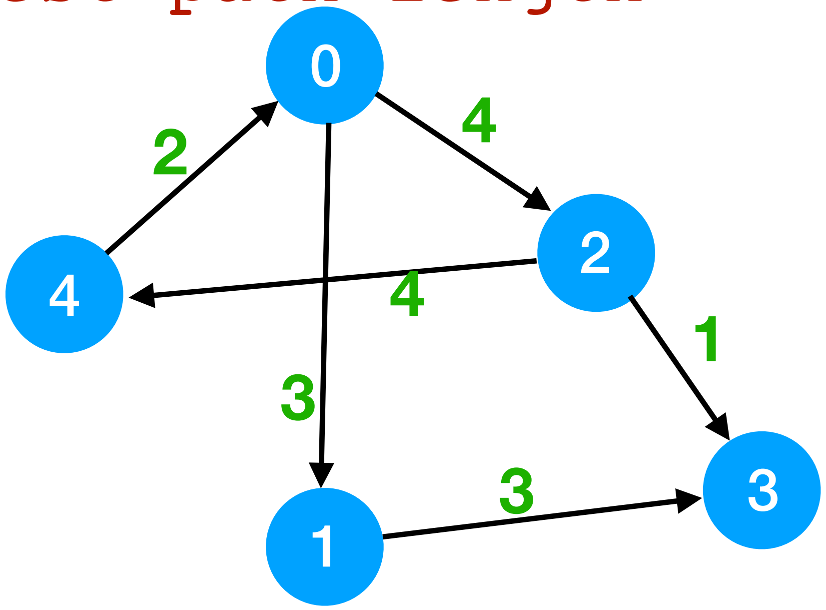
Node	d
0	2
1	?
2	?
3	?
4	0

```
Initialize Settled to empty
Initialize Frontier to the start node
While the frontier isn't empty:
  move the node f with smallest d from F to S
  For each neighbor w of f:
    if we've never seen w before:
      set its path length to f.d + wt(f,w)
      add w to the frontier
    else if the path to w via f is shorter:
      update w's shortest path length
```

f: 0

Settled set: {4, 0}

Frontier set: {}



shortest-paths(4)

Dijkstra's Shortest Paths: Execution

Best known distances:

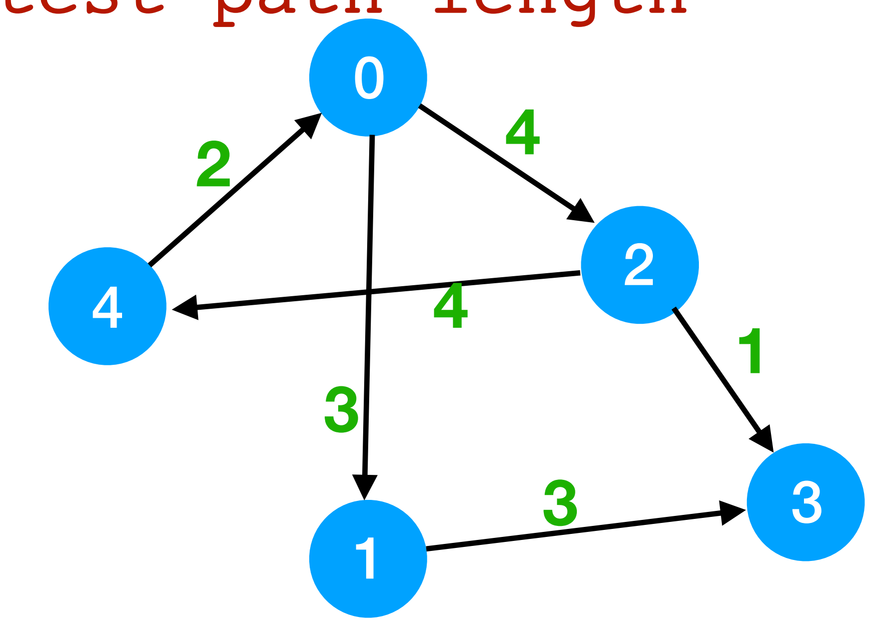
Node	d
0	2
1	**5**
2	?
3	?
4	0

```
Initialize Settled to empty
Initialize Frontier to the start node
While the frontier isn't empty:
  move the node f with smallest d from F to S
   For each neighbor w of f:
     if we've never seen w before:
       set its path length to f.d + wt(f,w)
       add w to the frontier
     else if the path to w via f is shorter:
       update w's shortest path length
```

f: 0
w: 1

Settled set: {4, 0}

Frontier set: {1}



shortest-paths(4)

Dijkstra's Shortest Paths: Execution

Best known distances:

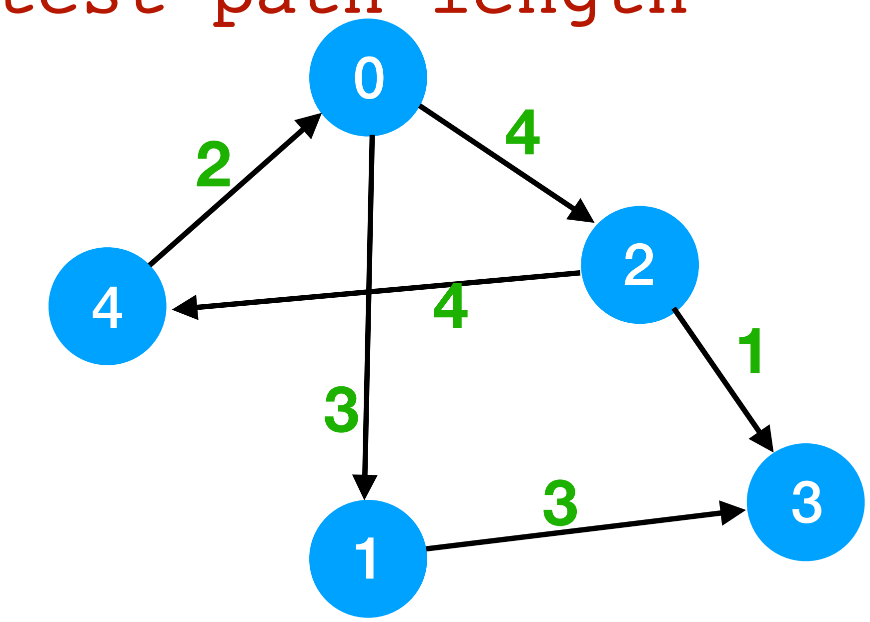
Node	d
0	2
1	5
2	**6**
3	?
4	0

```
Initialize Settled to empty
Initialize Frontier to the start node
While the frontier isn't empty:
  move the node f with smallest d from F to S
  For each neighbor w of f:
    if we've never seen w before:
      set its path length to f.d + wt(f,w)
      add w to the frontier
    else if the path to w via f is shorter:
      update w's shortest path length
```

f: 0
w: 2

Settled set: {4, 0}

Frontier set: {1, 2}



shortest-paths(4)

Dijkstra’s Shortest Paths: Execution

Best known distances:

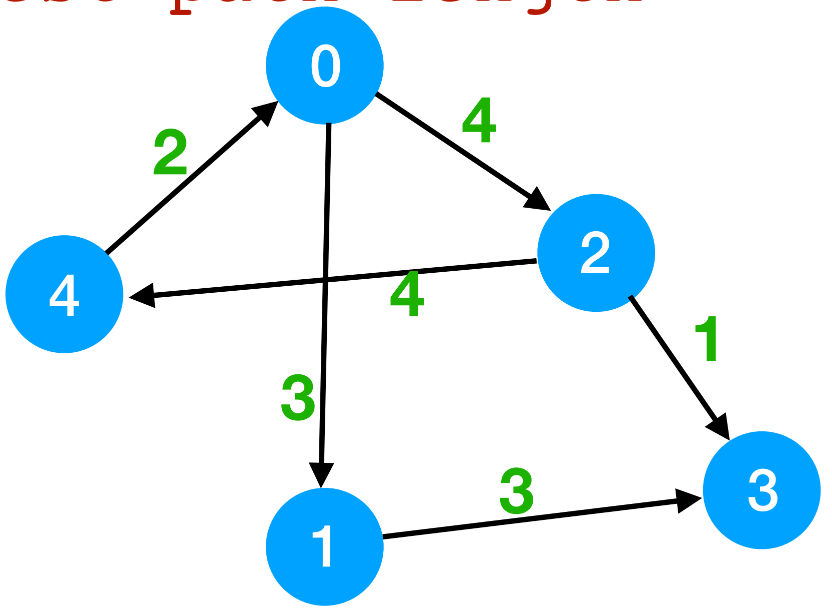
Node	d
0	2
1	5
2	6
3	8
4	0

```
Initialize Settled to empty
Initialize Frontier to the start node
While the frontier isn't empty:
  move the node f with smallest d from F to S
  For each neighbor w of f:
    if we've never seen w before:
      set its path length to f.d + wt(f,w)
      add w to the frontier
    else if the path to w via f is shorter:
      update w's shortest path length
```

f: 1

Settled set: {4, 0, 1}

Frontier set: {2}



shortest-paths(4)

Dijkstra’s Shortest Paths: Execution

Best known distances:

Node	d
0	2
1	5
2	6
3	8
4	0

```
Initialize Settled to empty
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While the frontier isn't empty:
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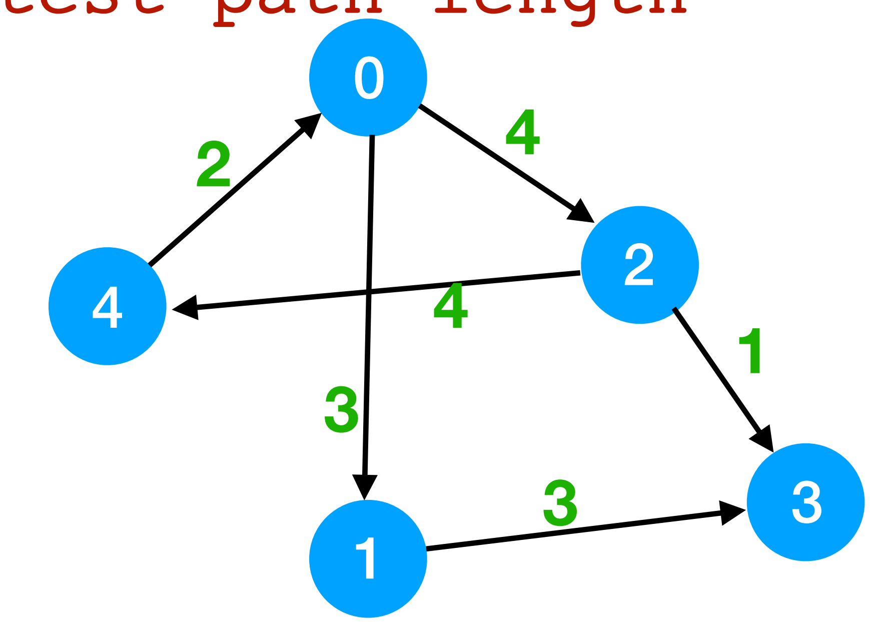
       add w to the frontier
     else if the path to w via f is shorter:
       update w's shortest path length
```

f: 1
w: 3

Settled set: {4, 0, 1}

Frontier set: {2, 3}



shortest-paths(4)

Dijkstra's Shortest Paths: Execution

Best known distances:

Node	d
0	2
1	5
2	6
3	8
4	0

```
Initialize Settled to empty
Initialize Frontier to the start node
While the frontier isn't empty:
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  For each neighbor w of f:
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    else if the path to w via f is shorter:
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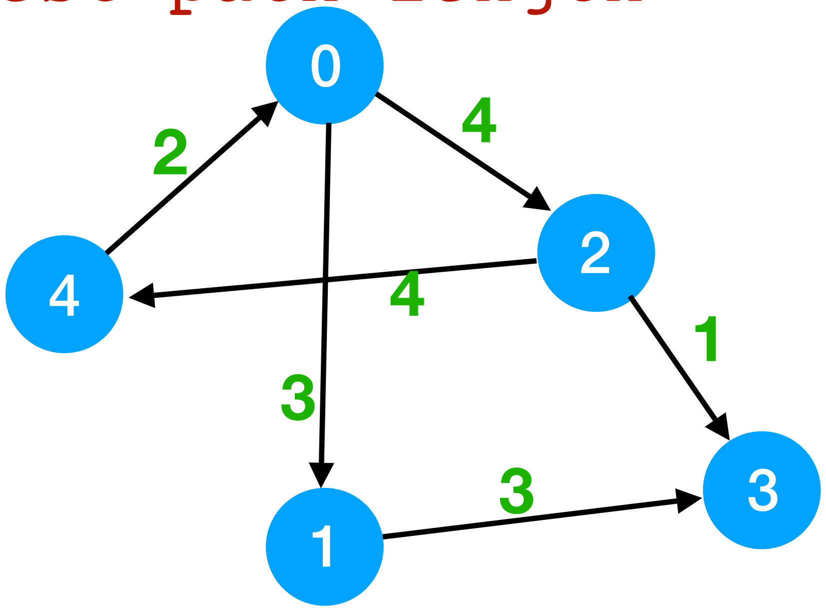
```

f: 2

Settled set: {4, 0, 1, 2}

Frontier set: {3}



shortest-paths(4)

Dijkstra's Shortest Paths: Execution

Best known distances:

Node	d
0	2
1	5
2	6
3	7
4	0

```
Initialize Settled to empty
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While the frontier isn't empty:
  move the node f with smallest d from F to S
  For each neighbor w of f:
    if we've never seen w before:
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      add w to the frontier
    else if the path to w via f is shorter:
      update w's shortest path length
```

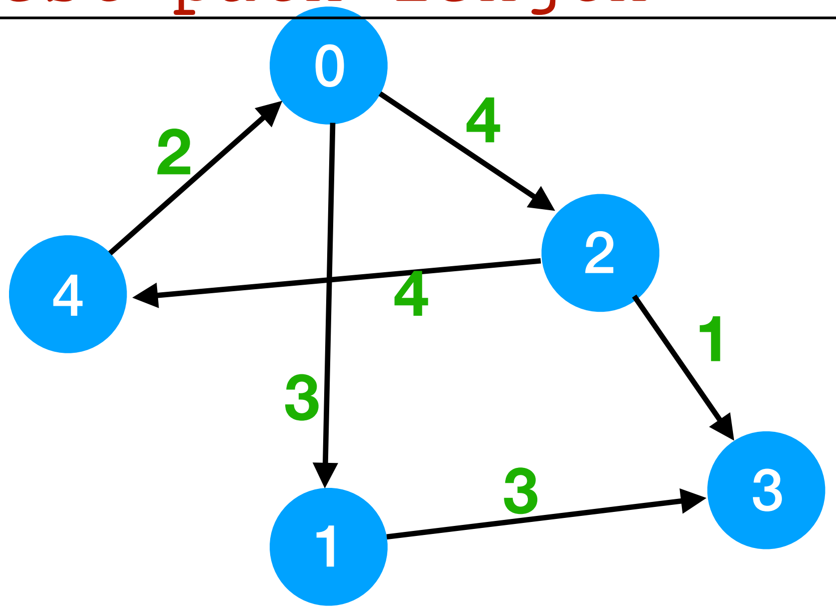
f: 2
w: 3

$2.d + wt(2,3) < 3.d$

$7 < 8$

Settled set: {4, 0, 1, 2}

Frontier set: {3}



`shortest-paths(4)`

Dijkstra's Shortest Paths: Execution

Best known distances:

Node	d
0	2
1	5
2	6
3	7
4	0

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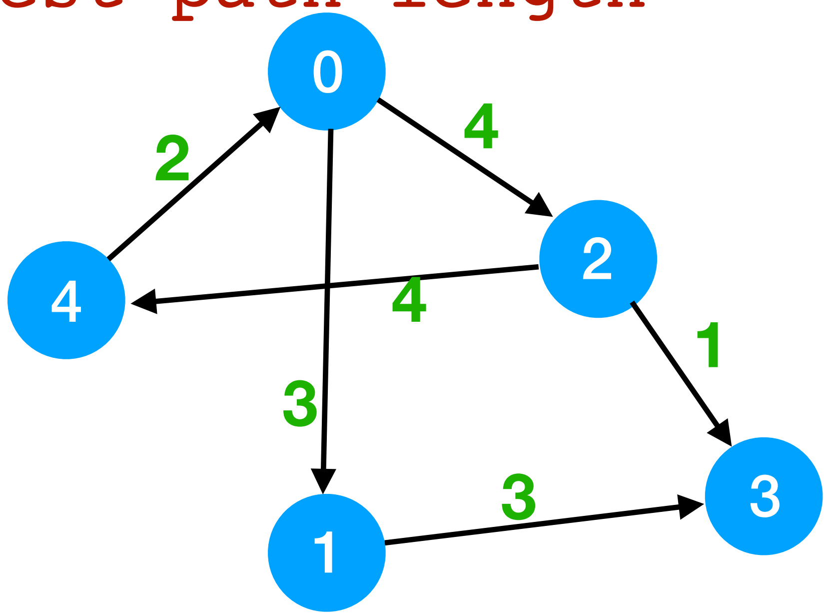
```

f: 3

Settled set: {4, 0, 1, 2, 3}

Frontier set: {}   Empty => done!



shortest-paths(4)

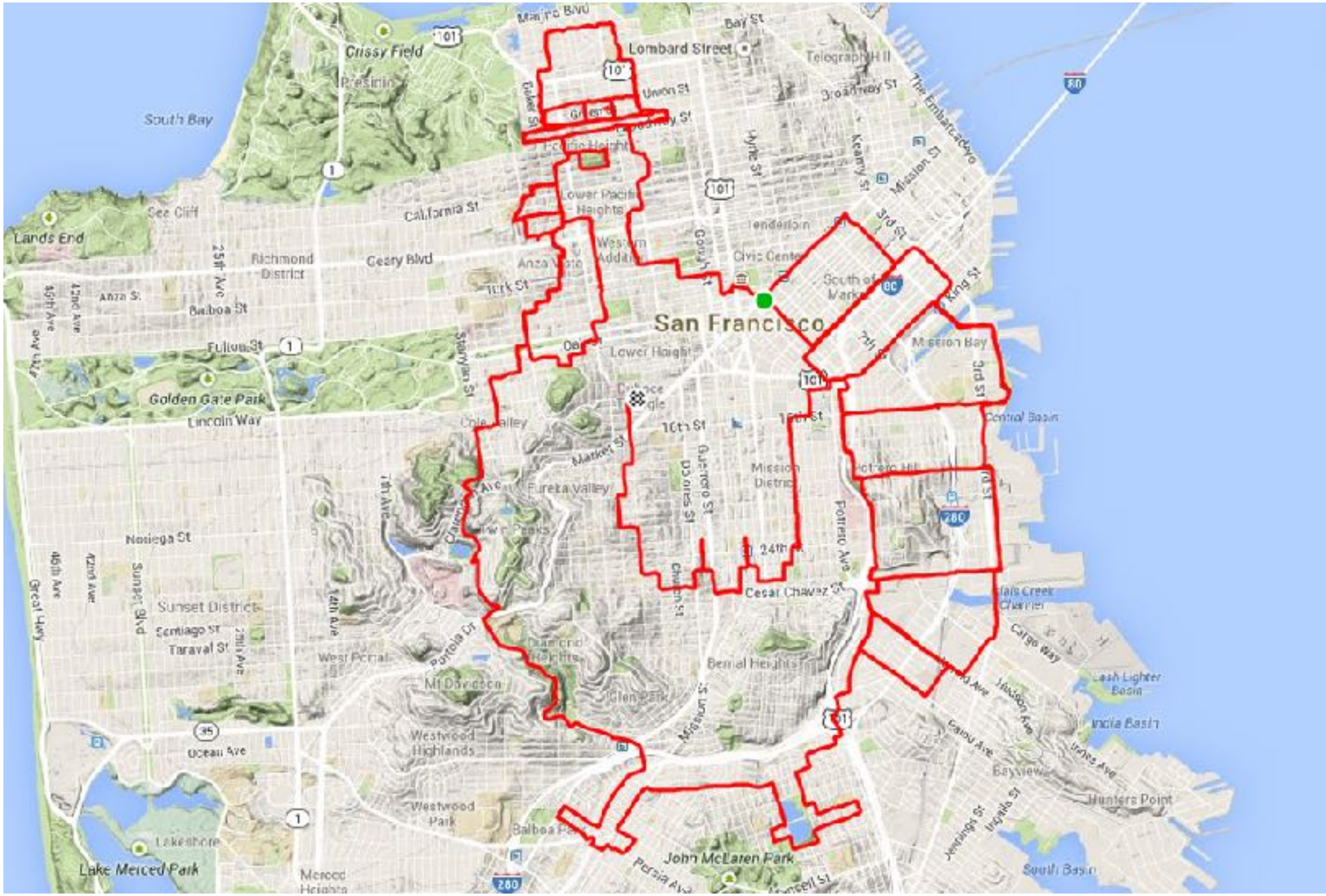
Let’s Dijkstra

- Half get the algorithm, half get the graphs.
- Run the algorithm on each graph:
    - (first) 5-node graph: start at node S
    - (second) Other graph: start at node D

Unanswered Questions

- Does this always work?
- How do you get the path, not just its length?
- How do you implement it efficiently?
- What’s the runtime?

Sometimes it's not about finding the shortest path.



Have a great Thanksgiving!