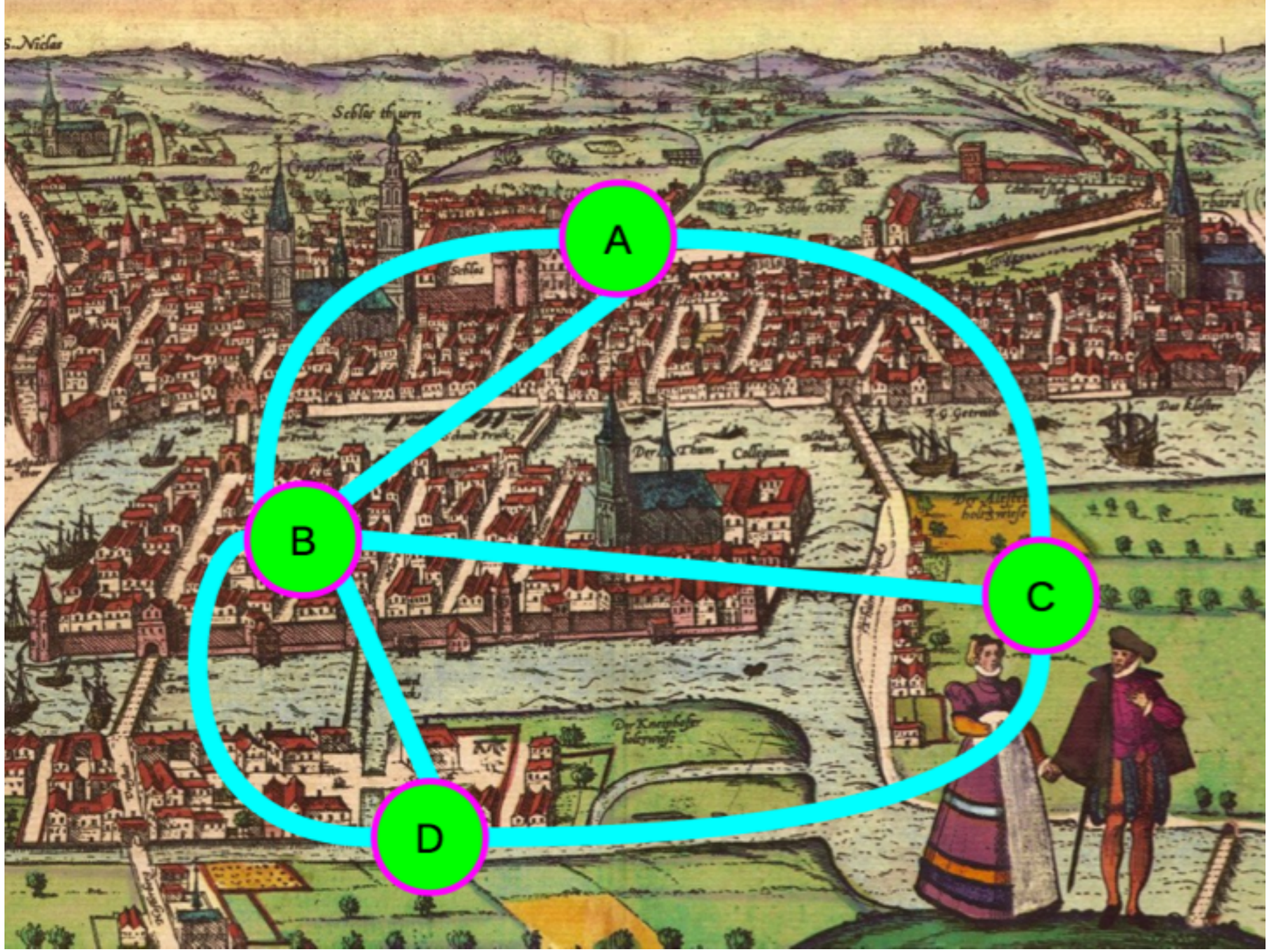


CSCI 241

Lecture 20

Graph Representations, Topological Sort

Announcements

- A3 has phases!
- They're all out now.
- My solution has 111 more lines of code than the skeleton.
- Don't procrastinate: There's a lot of conceptual stuff in those 111 lines.

A3 has 3 phases.

use AList to handle growing the array!

1. Write a min-heap to implement a priority queue with operations:
   - `boolean add(V value, P priority)`
   - `V peek();`
   - `V poll();`
2. Write a hash table.

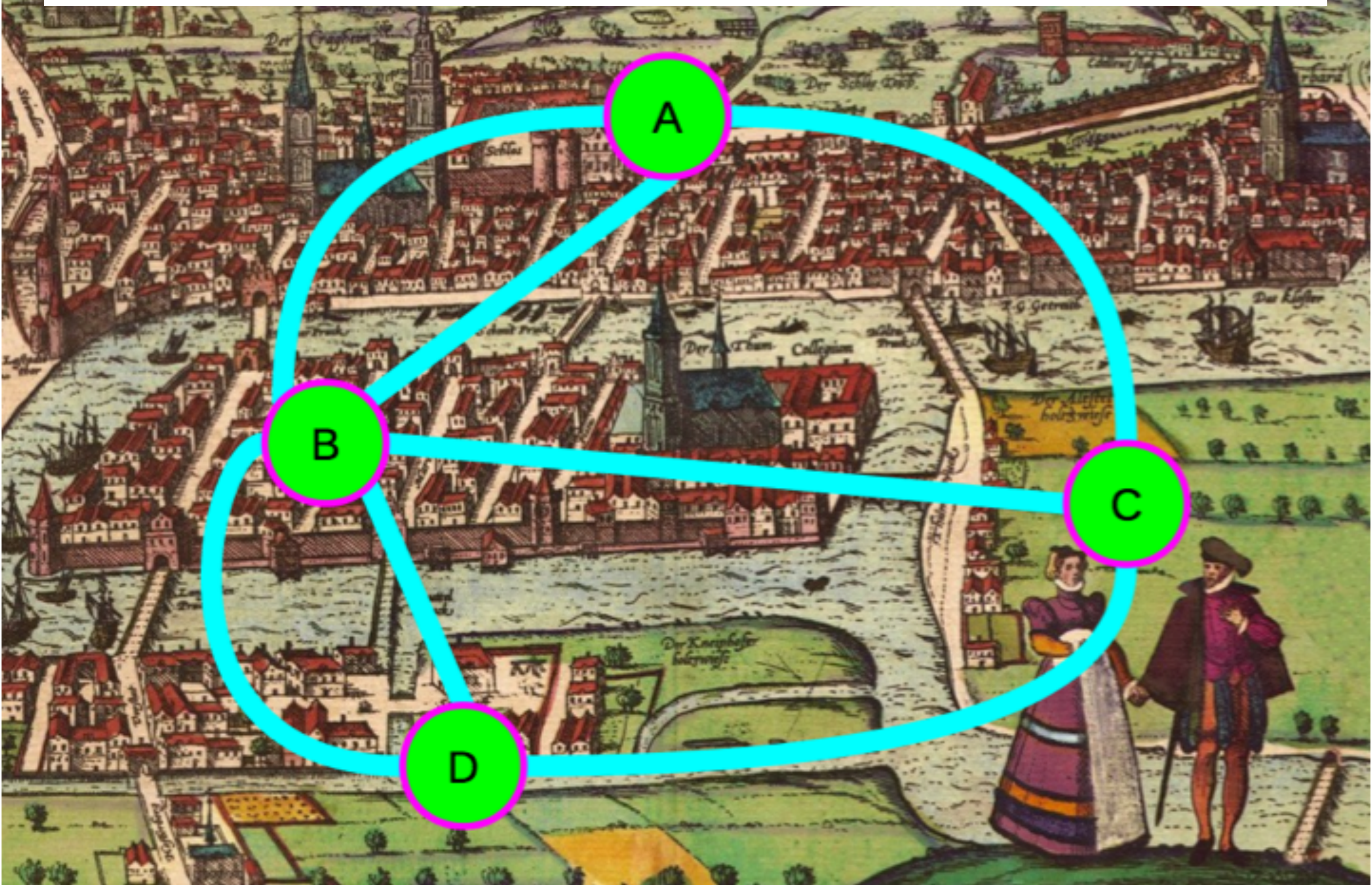
~~use AList to handle growing the array!~~

3. Use the hash table to augment the heap, making the following operations efficient:
   - `boolean contains(V v);`
   - `void changePriority(V v, P newP);`

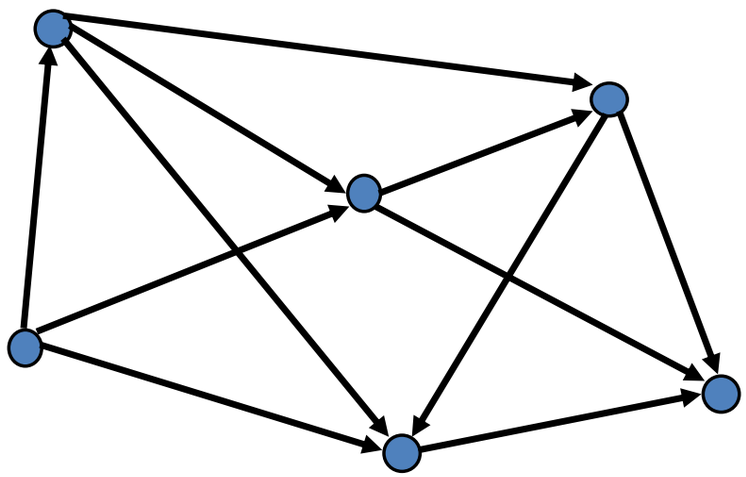
Goals

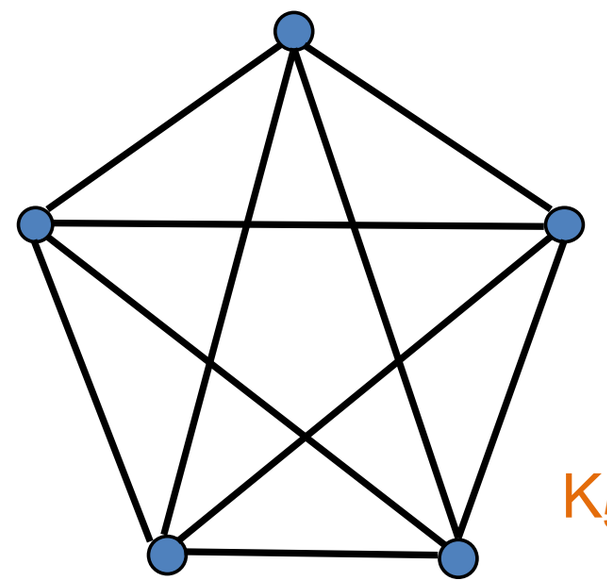
- Know some additional graph terminology:
  - Tree (graph definition), subgraph, planarity
- Understand how to represent a graph using:
  - adjacency list
  - adjacency matrix
- Be able to implement and analyze the runtime of simple graph operations on adjacency matrices and adjacency lists.
- Know how to detect whether a directed graph is acyclic using Topological Sort

Graph: a bunch of points connected by lines.
The lines may have directions, or not.

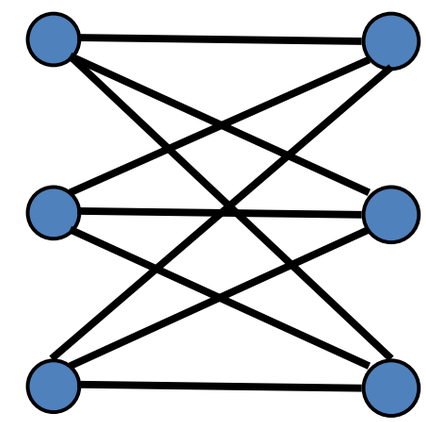


Graphs: Abstract View

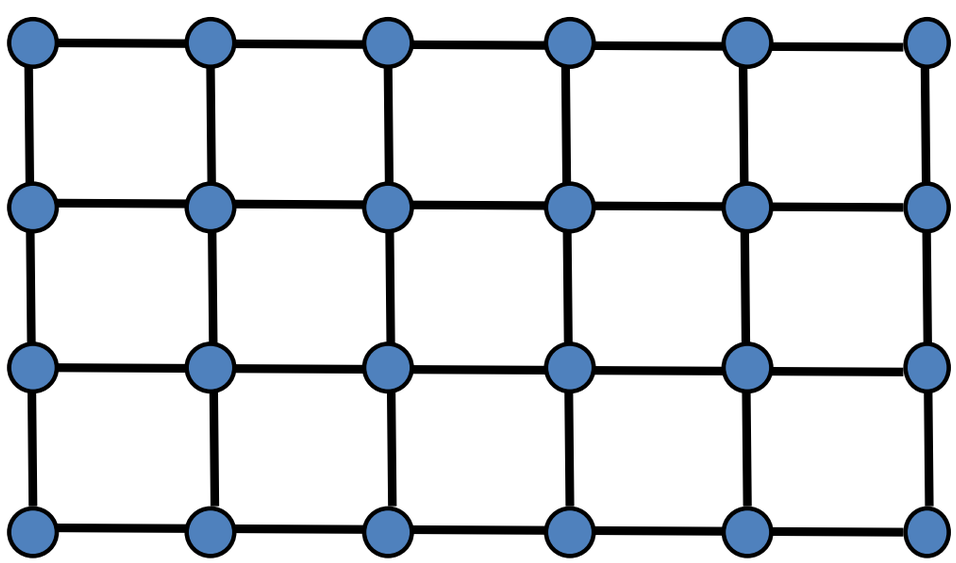


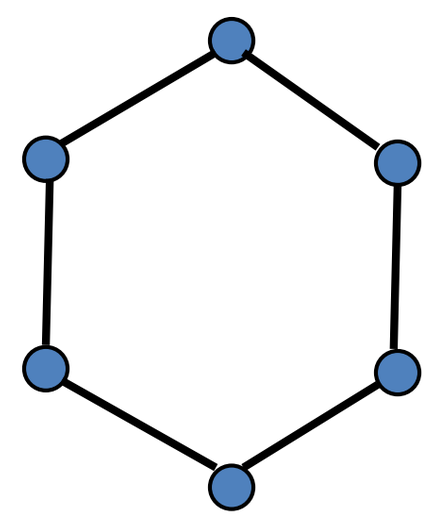


K_5



$K_{3,3}$



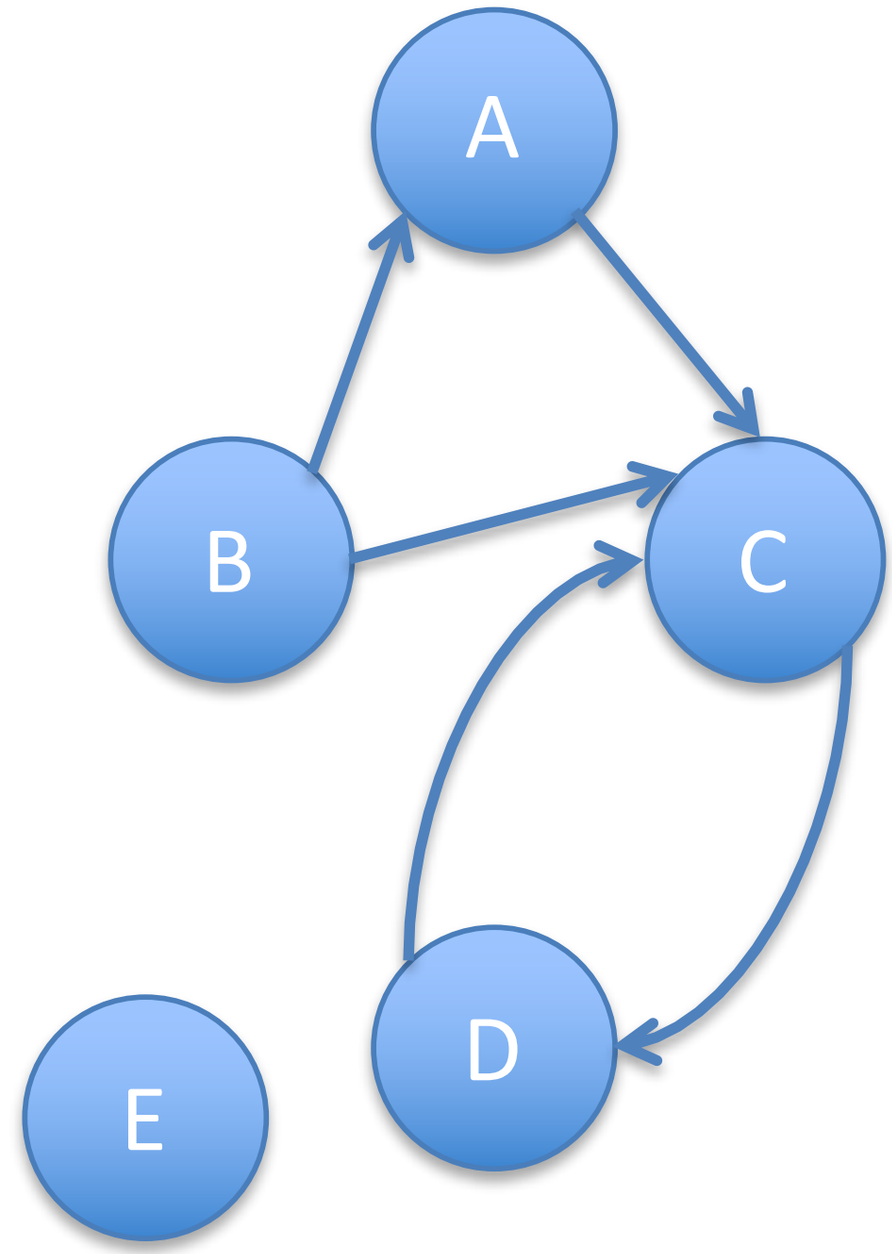


Graphs, Formally

- A directed graph (digraph) is a pair (V, E) where:
  - V is a (finite) set
  - E is a set of **ordered** pairs (u, v) where u, v are in V
  - Often (not always): $u \neq v$ (i.e. no edges from a vertex to itself)
- An element in V is called a vertex or node
- Elements in E are called edges or arcs
- $|V|$ = size of V (traditionally called n)
- $|E|$ = size of E (traditionally called m)

An example directed graph

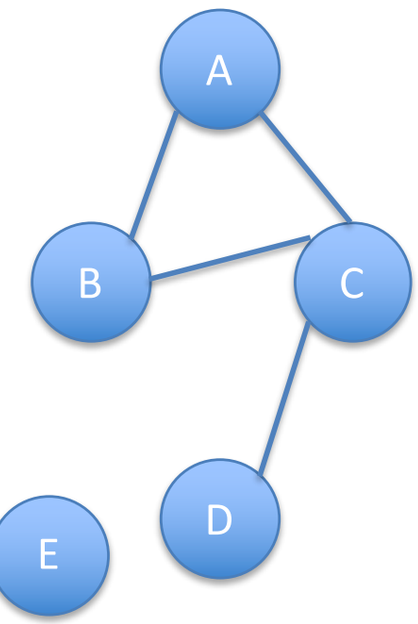
$$\boldsymbol{V} = \{A, B, C, D, E\}$$
$$\boldsymbol{E} = \{(A, C), (B, A), (B, C), (C, D), (D, C)\}$$
$$|\boldsymbol{V}| = 5$$
$$|\boldsymbol{E}| = 5$$



Graphs, Formally

- An **un**directed graph is a just like a digraph, but
  - E is a set of **un**ordered pairs (u, v) where u, v are in V

$$\boldsymbol{V} = \{A, B, C, D, E\}$$
$$\boldsymbol{E} = \{\{A,C\}, \{B,A\}, \{B,C\}, \{C,D\}\}$$
$$|\boldsymbol{V}| = 5$$
$$|\boldsymbol{E}| = 4$$



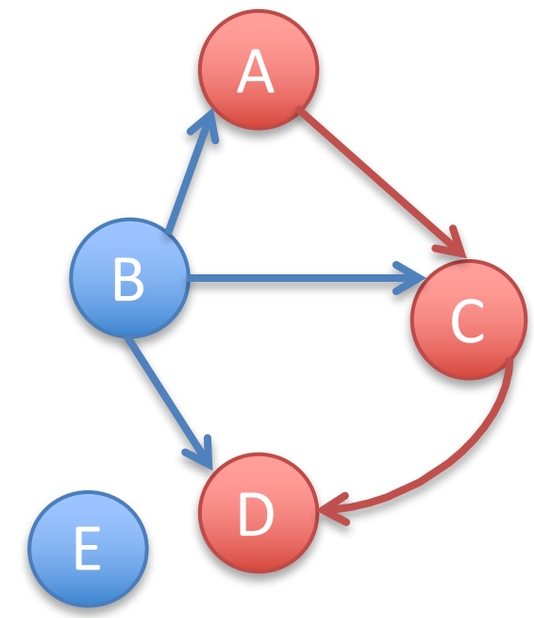
- An **un**directed graph can be converted to an equivalent **directed** graph:
  - Replace each undirected edge with two directed edges in opposite directions
- A **directed** graph can't always be converted to an **undirected** graph.

Graph Terminology: Adjacency, Degree

- Two vertices are adjacent if they are connected by an edge
- Nodes u and v are called the source and sink of the **directed** edge (u, v)
- Nodes u and v are endpoints of an edge (u, v) (directed or undirected)
- The outdegree of a vertex u in a **directed** graph is the number of edges for which u is the source
- The indegree of a vertex v in a **directed** graph is the number of edges for which v is the sink
- The degree of a vertex u in an **undirected** graph is the number of edges of which u is an endpoint

Graph Terminology

- A path is a sequence of vertices where each consecutive pair are adjacent.
- In a directed graph, paths must follow the direction of the edges.
- A cycle is a path that ends where it started, e.g.: x, y, z, x
- A graph is acyclic if it has no cycles.
- A graph is connected if there is a path between every pair of nodes.



Path A,C,D

Not connected

Acyclic

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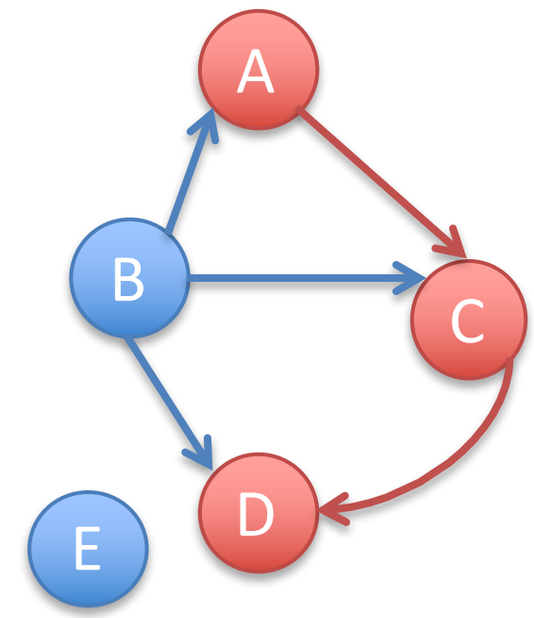


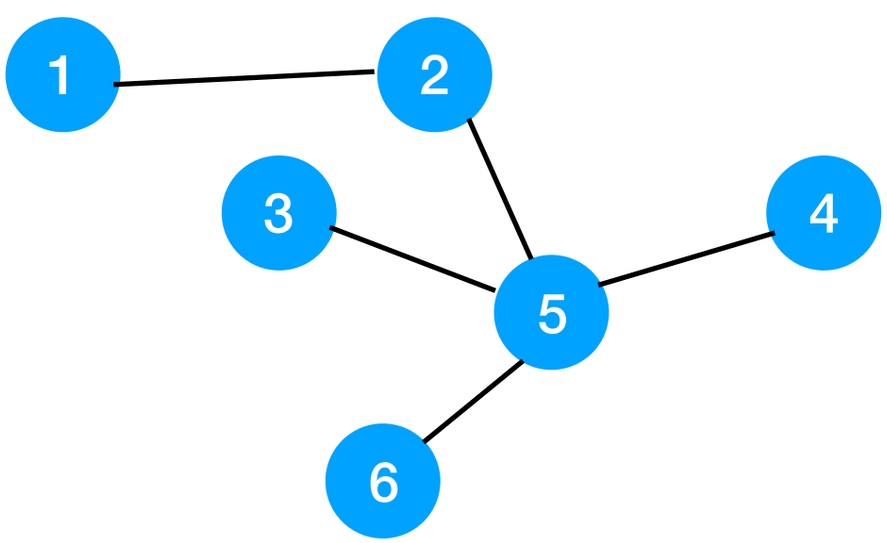
Path A,C,D

Not connected

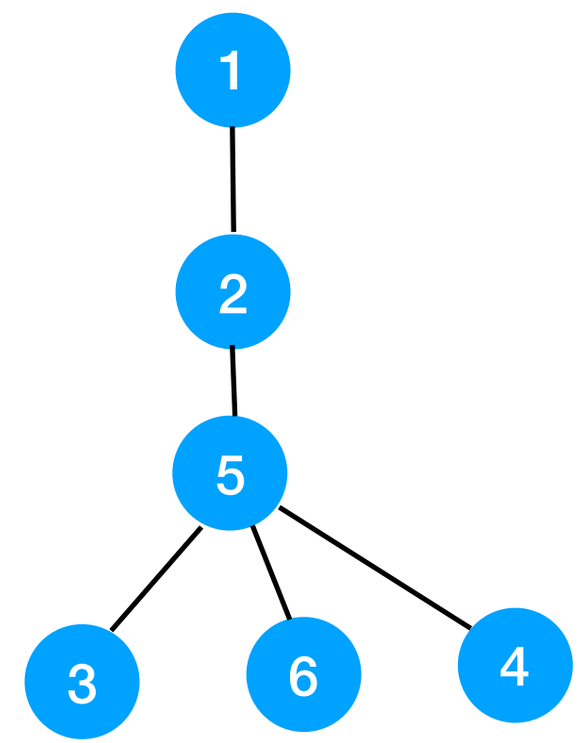
Acyclic

Trees vs Graphs

- Trees are graphs!
- A tree is an undirected graph with exactly 1 path between all pairs of nodes.
- Implication: no cycles!



V = {1,2,3,4,5,6}
E = {(1,2), (2,5), (3,5)
(4,5), (5,6)}



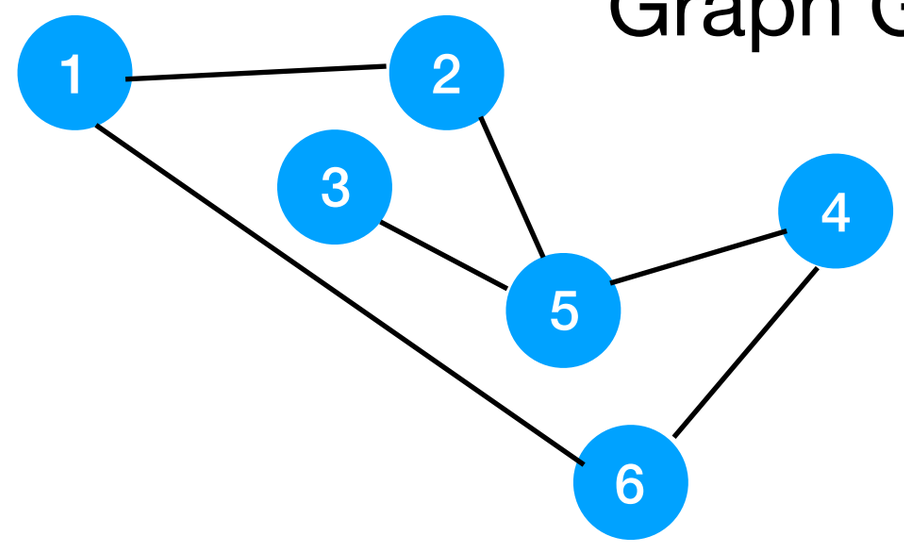
Many problems are easy in trees and harder in graphs.

Graph terminology: Lightning Round!

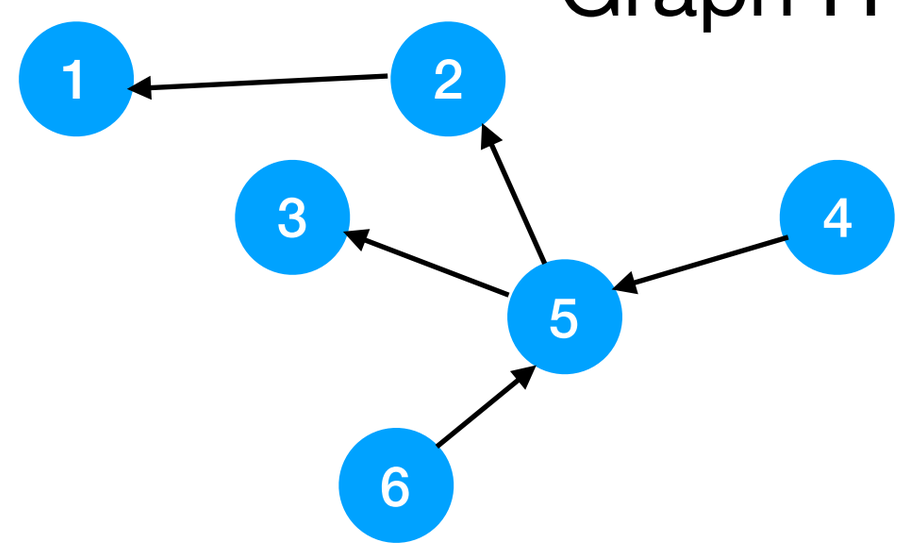
A: No B: Yes

- Is graph G acyclic?
- Is there a path from 3 to 5 in graph H?
- Is graph H directed?
- Is (1,2) an edge in H?

Graph G



Graph H



Graph terminology: Lightning Round!

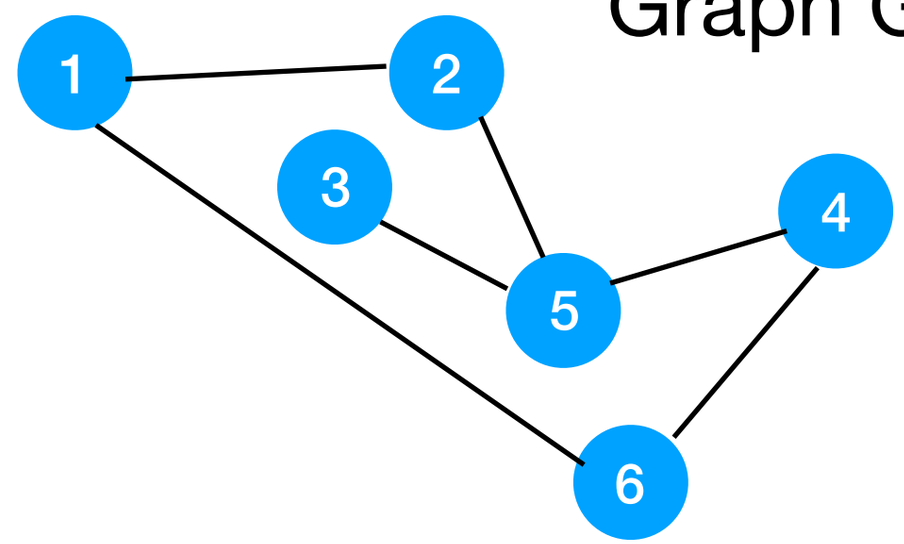
- What's the degree of node 5 in graph G?
  A: 1 B: 2 C: 3 D: 4

- What is |V| in graph G?
  A: 3 B: 4 C: 5 D: 6

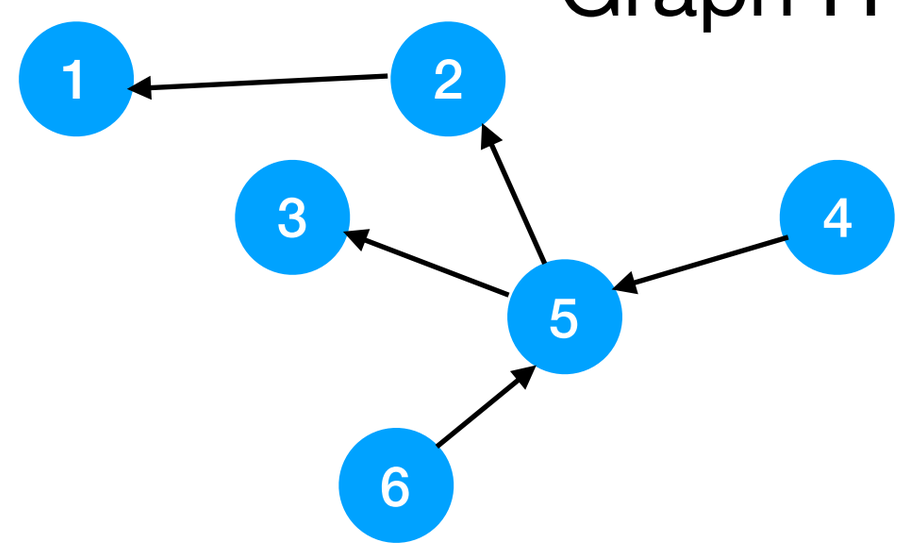
- What is |E| in graph H?
  A: 4 B: 5 C: 6 D: 7

- Is H connected?
  A: no B: yes

Graph G



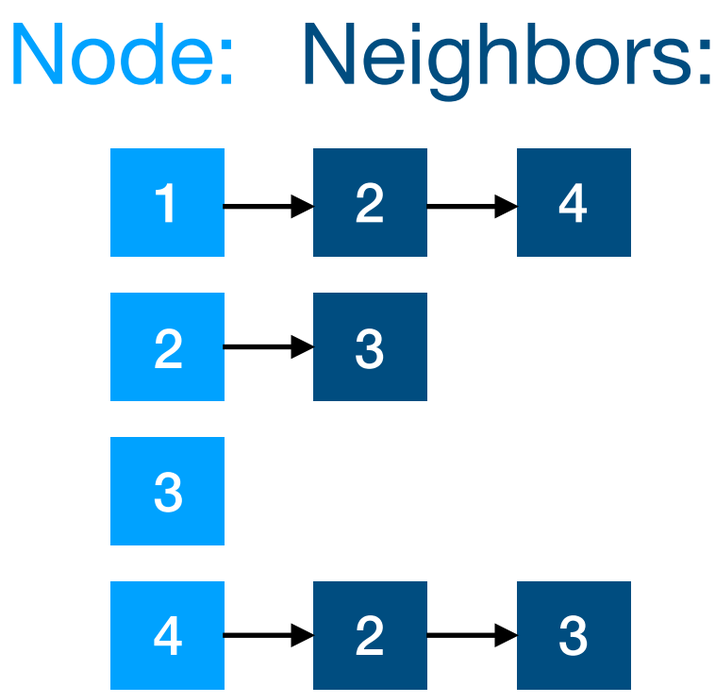
Graph H

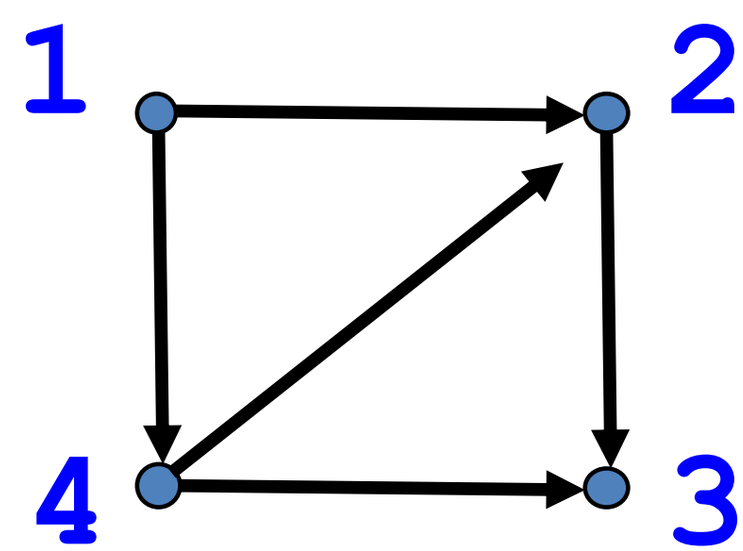


Representing Graphs: Adjacency Lists

```
public class GraphNode {
    // fields storing information
    // about this node

        List<GraphNode> neighbors;
  }
```



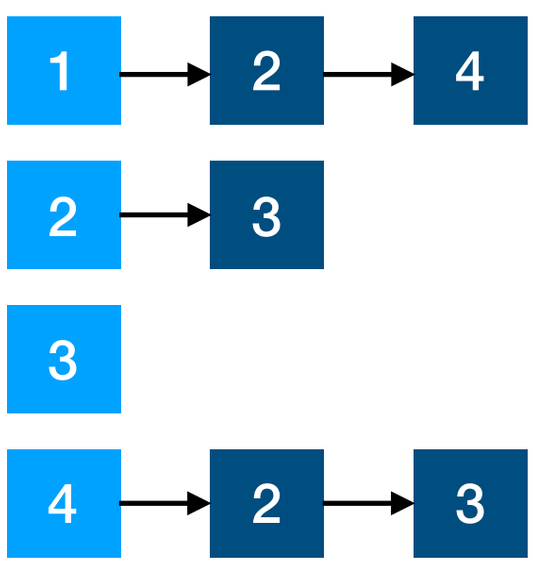


Representing Graphs: Adjacency Matrix

```
public class Graph {
    boolean[][] adjacent; // size n x n
}
```

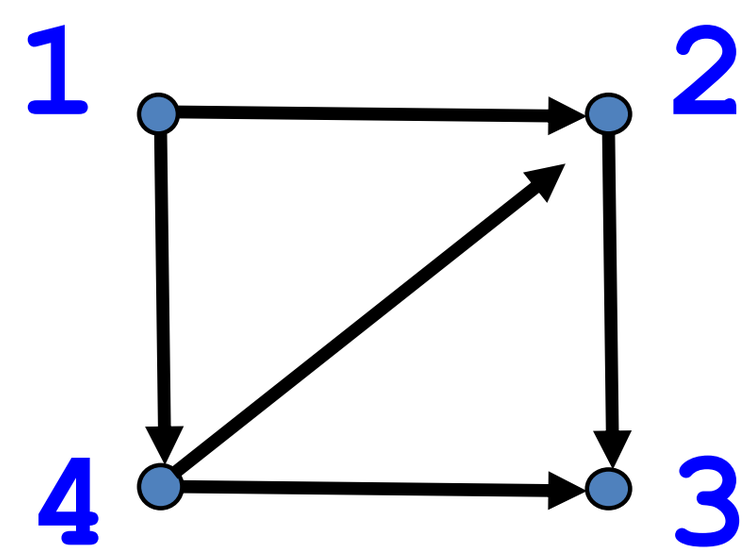
Adjacency lists:

Node: Neighbors:



Adjacency Matrix:

	1	2	3	4
1	0	1	0	1
2	0	0	1	0
3	0	0	0	0
4	0	1	1	0



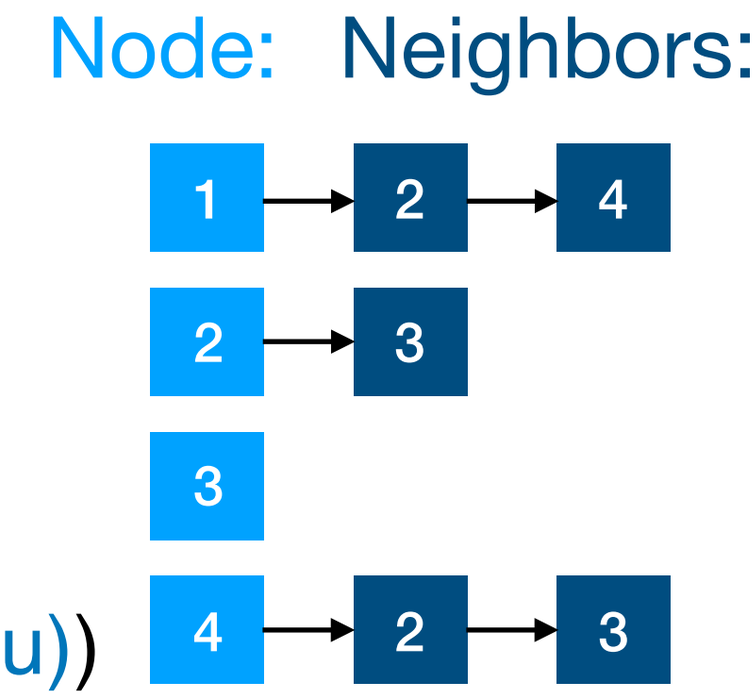
Adjacency Matrix vs Adjacency List

- Reminder: $n = |V|$ and $m = |E|$; let $d(u)$ = degree of u
- Adjacency matrix:
  - Storage space: $O(n^2)$
  - Iterate over edges: $O(n^2)$ time
  - Check if there's an edge from u to v: $O(1)$
  - Good for dense graphs
    - e.g., if $|E|$ is close to n^2, you need n^2 storage anyway.

	1	2	3	4
1	0	1	0	1
2	0	0	1	0
3	0	0	0	0
4	0	1	1	0

Adjacency Matrix vs Adjacency List

- Reminder: n = |V| and m = |E|; let d(u) = degree of u
- Adjacency list:
  - Storage space: O(n + e)
  - Iterate over edges: O(n + e) time
  - Check if there's an edge from u to v: O(d(u))
  - Good for more sparse graphs:
    - e.g., if |E| is close to n, n + e ~= 2n, which is O(n)



Graph Algorithms

You can take entire graduate-level courses on graph algorithms. In this class:

- Topological Sort: detect cycles in a directed graph
- Search/traversal: search for a particular node or traverse all nodes
  - Breadth-first
  - Depth-first
- Spanning trees
- Shortest Paths

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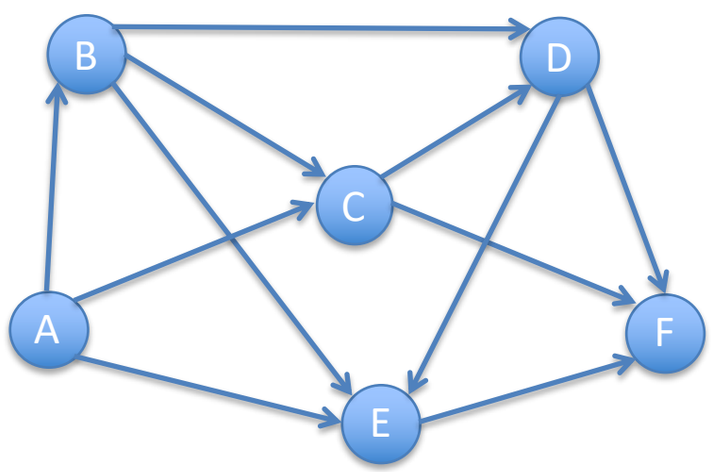
DAGs

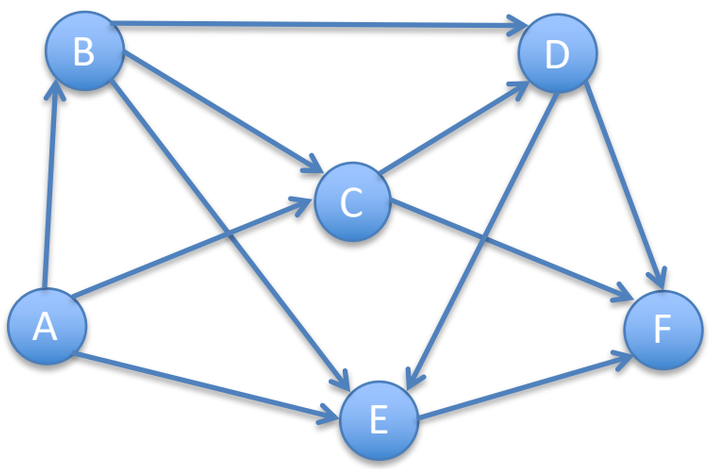
- A DAG, or Directed Acyclic Graph is a…

  graph that is directed and acyclic.



Is this a DAG?

- How do we tell if a directed graph is acyclic?
  - If a node has indegree 0, it can't be part of a cycle.
  - Edges coming from that node also can't be part of a cycle.



Algorithm:

while there is a node with indegree 0:

delete the node and all edges coming from it

if the graph is empty, the original graph was a DAG

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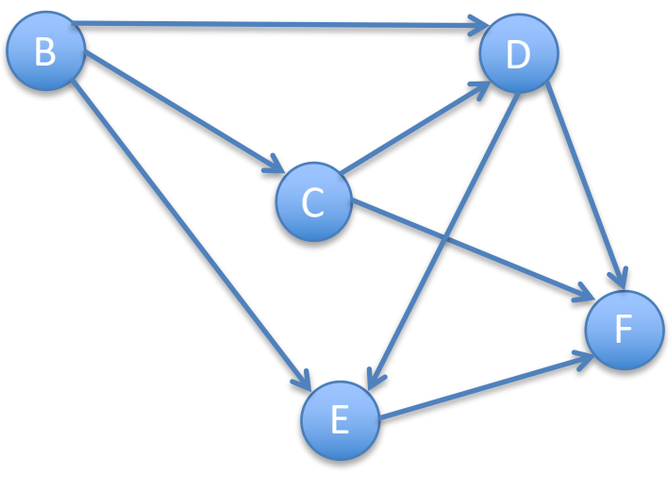

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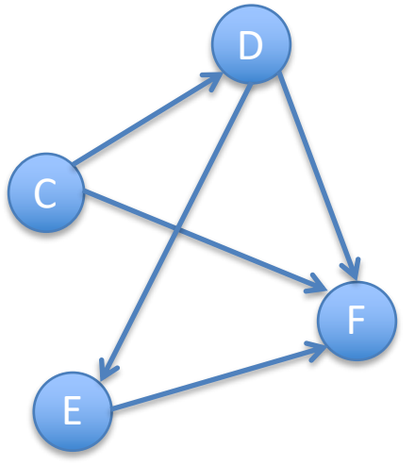

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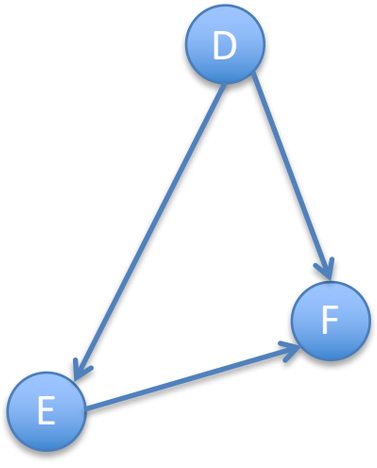

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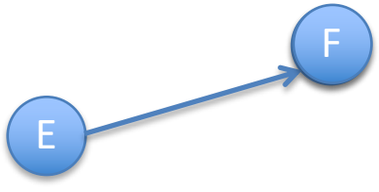

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Topological Sort

Topological sort (or toposort):

```
i = 0
while there is a node with indegree 0:
    delete* the node and all edges coming from it
    label* the deleted node i
    increment i
if the graph is empty, the original graph was a DAG
```

Topological Sort

Topological sort (or toposort):

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i = 0
while there is a node with indegree 0:
    delete* the node and all edges coming from it
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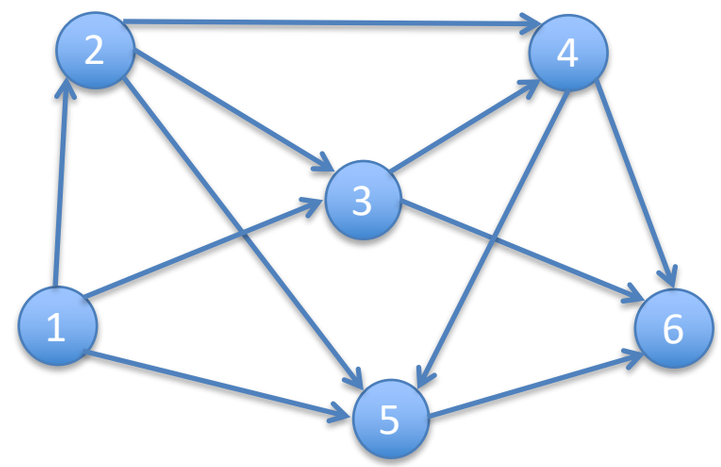
if the graph is empty, the original graph was a DAG
```

*This is pseudocode: we probably don't want to actually modify the graph. We'll need to store extra data with nodes and edges, and possibly overlay additional data structures to make it efficient.

Topological Sort

- Here are the labels we applied to the example graph:

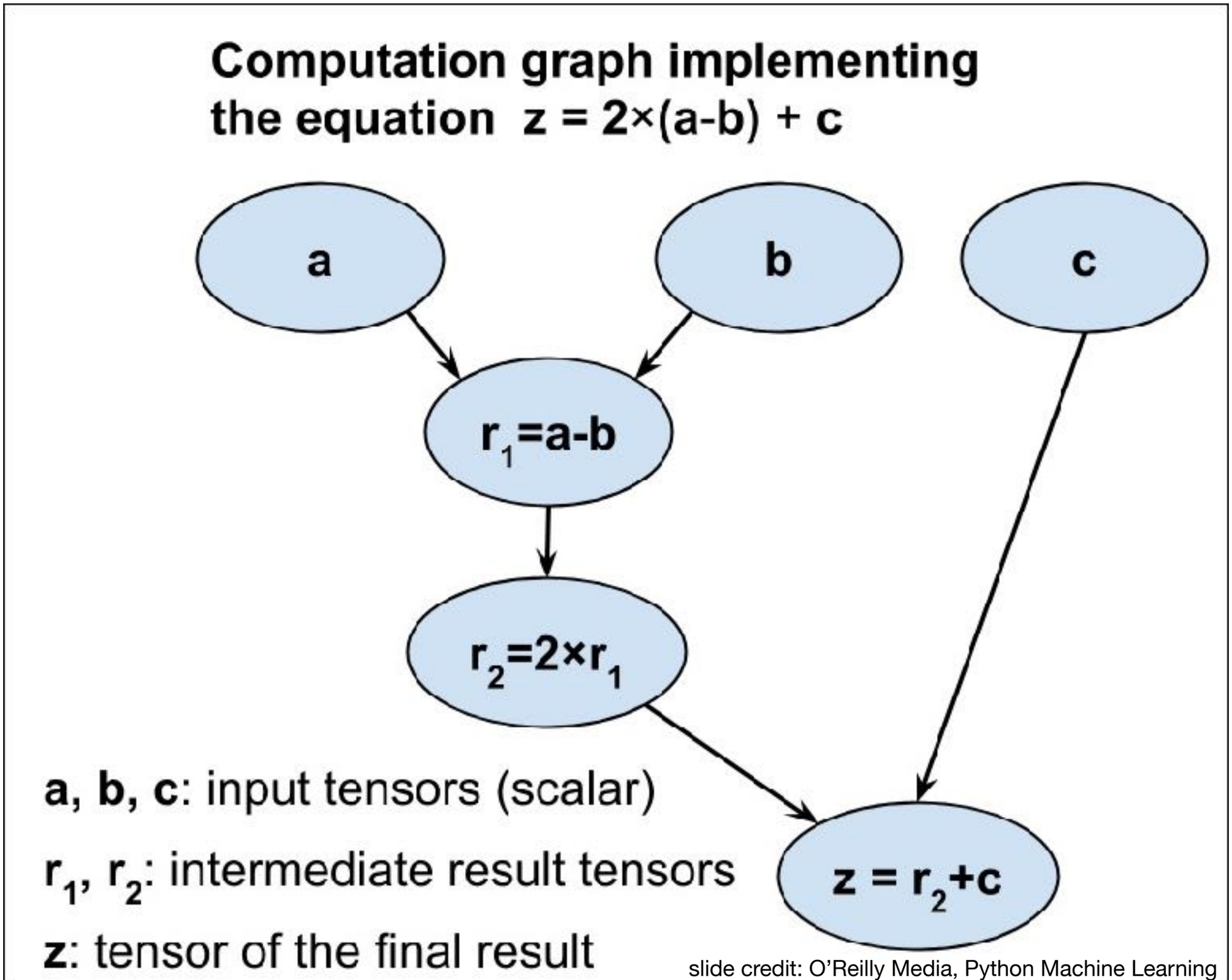


- Property: all edges go from a lower-numbered node to a higher-numbered node.

- Useful for dependency resolution, job scheduling,

- Ordering is not necessarily unique: could have chosen from among multiple nodes with indegree 0.

Tensorflow Computation Graphs

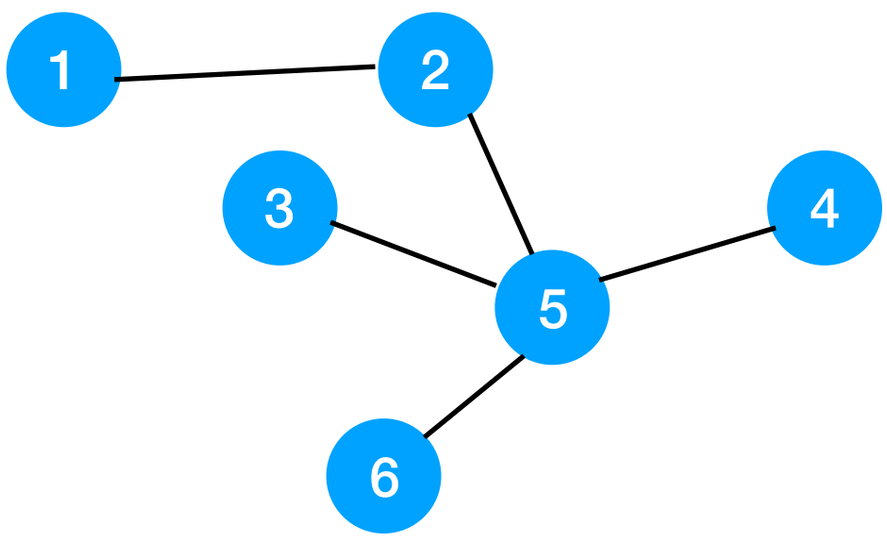


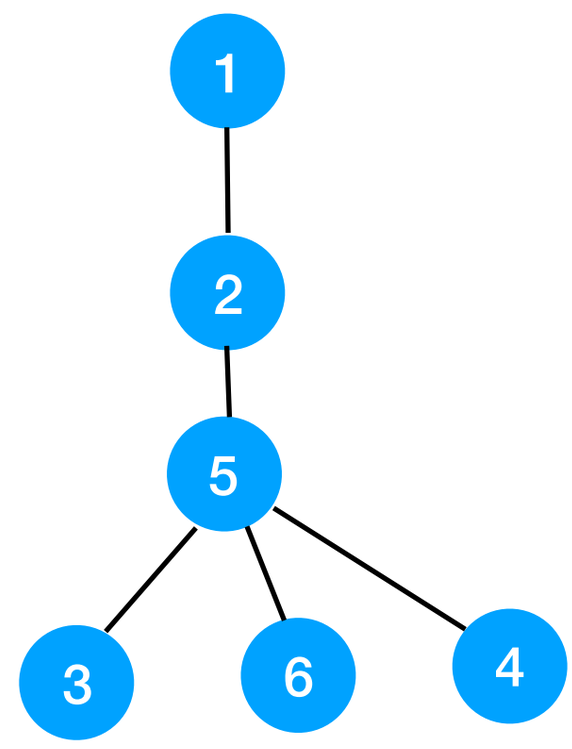
slide credit: O'Reilly Media, Python Machine Learning

Drawing Graphs

- The same graph can be drawn (infinitely!) many different ways.

$$V = \{1,2,3,4,5,6\}$$
$$E = \{(1,2), (2,5), (3,5)$$
$$(4,5), (5,6)\}$$

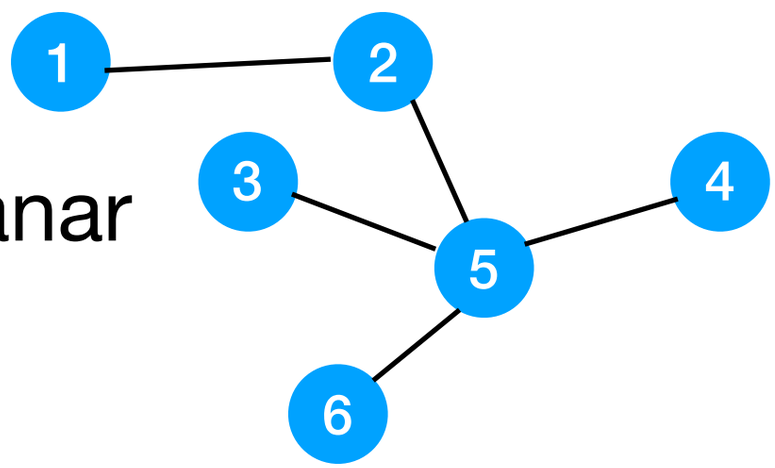


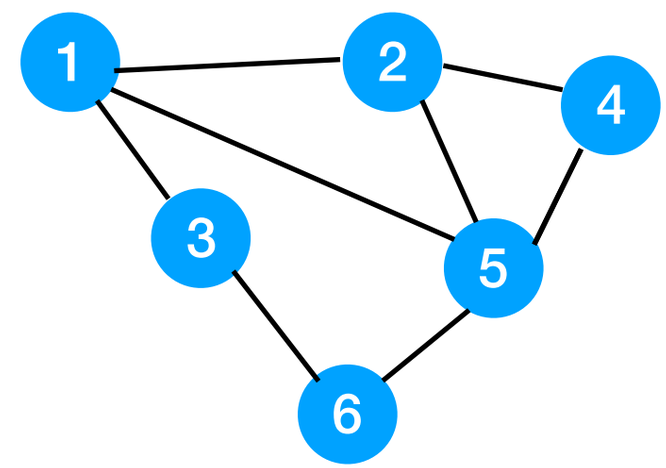


Planarity

- If a graph can be drawn without crossing edges, it is planar.

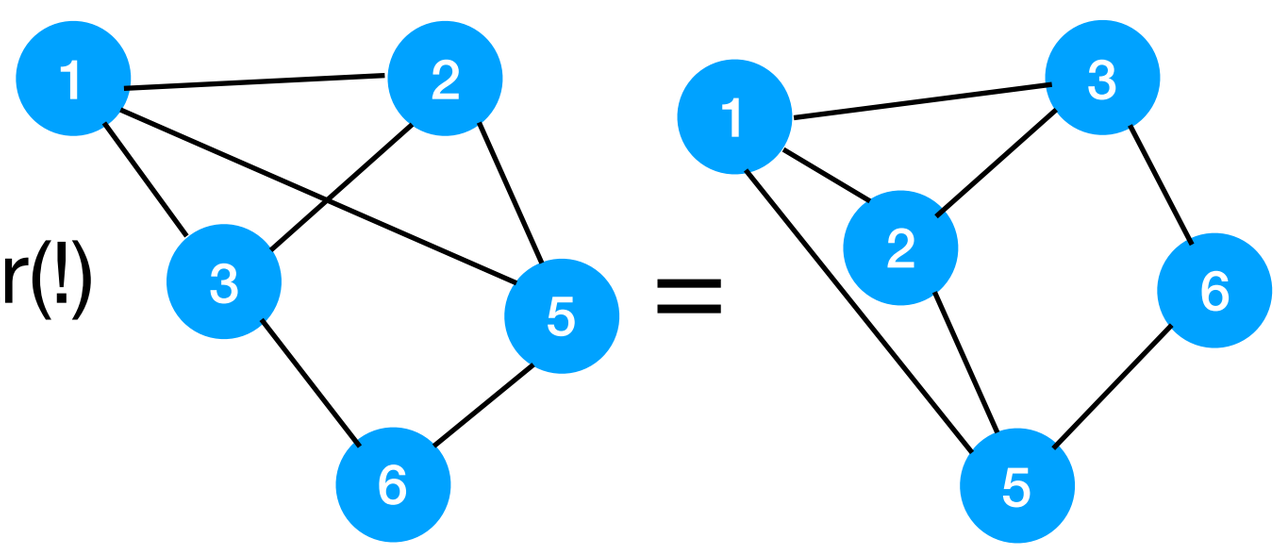
Planar

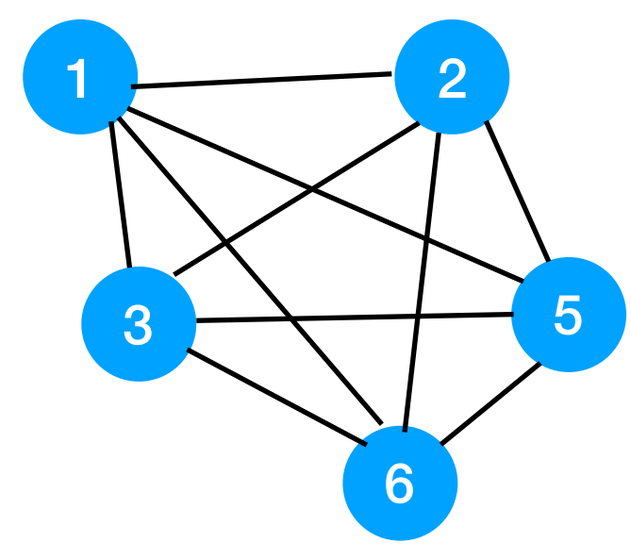




Planar

Planar(!)

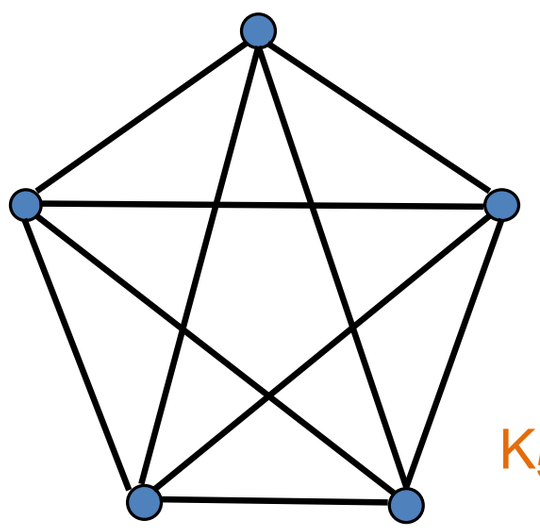




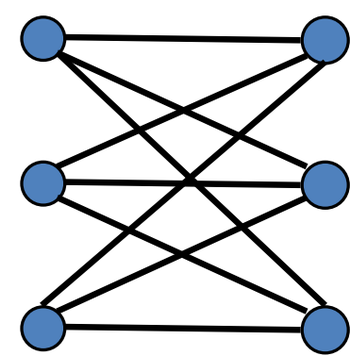
Not planar

Detecting Planarity

- It's tricky! There's a theorem that says a graph is planar if and only if it contains one of these as a subgraph:



K_5



$K_{3,3}$

A subgraph of a graph is a graph whose vertex and edge sets are subsets of the larger graph's.

- Elements of the edge subset can only contain nodes in the vertex subset.

Midterm

Grades will be released on Gradescope and Canvas after class today.

Exam is out of 50 points.
Guidelines for interpretation:

43+: A range

35+: B range

<35: C range

We're not perfect. Check your exam for grading errors!