Happenings

Happenings this week:
• Tuesday, Oct. 23 – Peer Lecture Series: Command Line Basics – 4 pm in CF 420
• Wednesday, Oct. 24 – Tech Talk: Accenture – 5 pm in CF 125
• Thursday, Oct. 25 – Group Advising to Declare the CS Major! – 3 pm in CF 420
• Friday, Oct. 26 – Cybersecurity Internship Workshop – 12 pm in AW 203
Goals

• Be prepared to implement AVL rebalancing.
Announcements

• Quiz 3 is not graded yet =( 

• A1 will be out tonight. It will be due Monday 10/29.

• How to see your Quiz 2:

  1. Go to www.gradescope.com

  2. Click Log In (not Sign Up - your account exists already)

  3. Click “Forgot your password?”, enter your WWU email address.

  4. Set your password using the link you receive by email.

  5. Log into gradescope.com using this password.
Where we left off on Friday
Where we left off on Friday
What are we even doing

- Trees
- Binary Trees
- Binary Search Trees
- Balanced BSTs
- AVL trees
What are we even doing

- Trees  what are they? nodes with 0 or more children (subtrees)
- Binary Trees  nodes with 0, 1, or 2 children (subtrees)
- Binary Search Trees  search, insert, remove and their runtimes
- Balanced BSTs  why do we want this? balanced trees give good runtime
- AVL trees  how do we measure balance? balance factor
  how do we achieve balance? rotations
  how do we know what/when to rotate?
Heights and Balance Factors

**Height**\((t)\): path length from \(t\)’s deepest descendant (leaf) to \(t\)’s root.

**Height**\((n)\): height of the subtree rooted at \(n\)

**Balance**\((n)\): \(\text{height}(n.\text{right}) - \text{height}(n.\text{left})\)

\[
\text{height}(null) = -1
\]

\[
\text{height}(n) = 1 + \max(\text{height}(n.\text{left}), \text{height}(n.\text{right}))
\]
Heights and Balance Factors

Height(t): path length from t’s deepest descendant (leaf) to t’s root.

Height(n): height of the subtree rooted at n

Balance(n): \( \text{height}(n.\text{right}) - \text{height}(n.\text{left}) \)

\[
\text{height}(\text{null}) = -1 \\
\text{height}(n) = 1 + \max(\text{height}(n.\text{left}), \text{height}(n.\text{right}))
\]
Heights and Balance Factors

Height(t): path length from t’s deepest descendant (leaf) to t’s root.

Height(n): height of the subtree rooted at n

Balance(n): height(n.right) - height(n.left)

height(null) = \(-1\)
height(n) = \(1 + \max(\text{height}(n.\text{left}), \text{height}(n.\text{right}))\)
Heights and Balance Factors

**Height**($t$): path length from $t$’s deepest descendant (leaf) to $t$’s root.

**Height**($n$): height of the subtree rooted at $n$

**Balance**($n$): $\text{height}(n.\text{right}) - \text{height}(n.\text{left})$

$\text{height}(\text{null}) = -1$

$\text{height}(n) = 1 + \max(\text{height}(n.\text{left}), \text{height}(n.\text{right}))$
Heights and Balance Factors

Height(t): path length from t’s deepest descendant (leaf) to t’s root.

Height(n): height of the subtree rooted at n

Balance(n): \( \text{height}(n.\text{right}) - \text{height}(n.\text{left}) \)

\[
\begin{align*}
\text{height}(\text{null}) &= -1 \\
\text{height}(n) &= 1 + \max(\text{height}(n.\text{left}), \text{height}(n.\text{right}))
\end{align*}
\]
Heights and Balance Factors

**Height**\( (t) \): path length from \( t \)'s deepest descendant (leaf) to \( t \)'s root.

**Height**\( (n) \): height of the subtree rooted at \( n \)

**Balance**\( (n) \): \( \text{height}(n.\text{right}) - \text{height}(n.\text{left}) \)

\[
\text{height}(\text{null}) = -1 \\
\text{height}(n) = 1 + \max(\text{height}(n.\text{left}), \text{height}(n.\text{right}))
\]
Heights and Balance Factors

**Height(t):** path length from t’s deepest descendant (leaf) to t’s root.

**Height(n):** height of the subtree rooted at n

**Balance(n):** $\text{height}(n.\text{right}) - \text{height}(n.\text{left})$

height(null) = -1

height(n) = 1 + max(height(n.left), height(n.right))
Heights and Balance Factors

Height(t): path length from t’s deepest descendant (leaf) to t’s root.

Height(n): height of the subtree rooted at n

Balance(n): $\text{height}(n.\text{right}) - \text{height}(n.\text{left})$

height(null) = -1

height(n) = 1 + \text{max}(\text{height}(n.\text{left}), \text{height}(n.\text{right}))
Heights and Balance Factors

Height(t): path length from t’s deepest descendant (leaf) to t’s root.

Height(n): height of the subtree rooted at n

Balance(n): height(n.right) - height(n.left)

height(null) = -1
height(n) = 1 + max(height(n.left), height(n.right))
Heights and Balance Factors

Height(t): path length from t’s deepest descendant (leaf) to t’s root.

Height(n): height of the subtree rooted at n

Balance(n): height(n.right) - height(n.left)

height(null) = -1
height(n) = 1 + max(height(n.left), height(n.right))
Tree Rotations

Steps in left rotation (move y up to x’s position):
1. Transfer β
2. Transfer the parent
3. Transfer x itself

x.R gets y.L
y.L.p gets x
y.p gets x.p
p.[L/R] gets y
y.L gets x
x.p gets y

Right rotation

Left rotation
Can we improve balance?

Balance(n): \text{height}(n.\text{right}) - \text{height}(n.\text{left})
Can we improve balance?

Balance(n): height(n.right) - height(n.left)
Can we improve balance?

**Balance**\( (n) \): \( \text{height}(n.\text{right}) - \text{height}(n.\text{left}) \)

\[
\begin{array}{c}
\text{8} & \text{10} \\
\text{8} & \text{11} & \text{15} \\
\text{-1} & \text{9} & \text{16} \\
\end{array}
\quad \rightarrow \quad
\begin{array}{c}
\text{8} & \text{10} & \text{15} \\
\text{8} & \text{9} & \text{11} & \text{16} \\
\text{-1} & \text{9} & \text{11} & \text{16} \\
\end{array}
\]

**ABCD**: The largest balance factor in the tree is:

A. 0 before, 1 after rotation
B. 1 before, 0 after rotation
C. 0 before, 2 after rotation
D. 2 before, 1 after rotation
Can we improve balance?

Balance(n): \( \text{height}(n.\text{right}) - \text{height}(n.\text{left}) \)

**ABCD**: The largest (absolute) balance factor in the tree is:

A. 0 before, 1 after rotation
B. 1 before, 0 after rotation
C. 0 before, 2 after rotation
D. 2 before, 0 after rotation
Can we improve balance?

Balance(n): height(n.right) - height(n.left)

How do we know what to rotate and when?
If the tree changes, check for imbalance and fix it if found.
AVL Trees

Balance(n): height(n.right) - height(n.left)

- Devised by Adelson-Velsky and Landis
- An AVL tree is a Binary Search Tree in which the following property holds:

**AVL property:** -1 <= balance(n) <= 1 for all nodes n.
Balance Factor in AVL Trees

**AVL property:** \(-1 \leq \text{balance}(n) \leq 1\) for all nodes \(n\).

Every subtree in an AVL tree looks like one of these three trees:

(a) Balance factor: 1

(b) Balance factor: 0

(c) Balance factor: -1
AVL Trees: Insertion

- An AVL tree is a Binary Search Tree in which the following property holds:

**AVL property:** \(-1 \leq b(n) \leq 1\) for all nodes \(n\).

To insert into an AVL tree:
1. Do a normal BST insertion
2. Fix any violations of the AVL property using rotations.

(a) Balance factor: 1  
(b) Balance factor: 0  
(c) Balance factor: -1
AVL Trees: Insertion

**AVL property**: $-1 \leq b(n) \leq 1$ for all nodes $n$.

To insert into an AVL tree:
1. Do a normal BST insertion
2. Fix any violations of the AVL property using rotations.

(a) Balance factor: 1  
(b) Balance factor: 0  
(c) Balance factor: -1
Refresher: BST Insertion

/* insert a node with value v into the * tree rooted at n. pre: n is not null. */
insert(Node n, int v):
  if n.value == v: return // (duplicate)
  if v < n.value:
    if n has left:
      insert(n.left, v)
    else:
      // attach new node w/ value v to n.left
  else:
    // v > n.value
    if n has right:
      insert(n.right, v)
    else:
      // attach new node w/ value v to n.right
AVL Insertion

/* insert a node with value v into the
* tree rooted at n. pre: n is not null. */
insert(Node n, int v):
    if n.value == v: return // (duplicate)
    if v < n.value:
        if n has left:
            insert(n.left, v)
        else:
            // attach new node w/ value v to n.left
    else: // v > n.value
        if n has right:
            insert(n.right, v)
        else:
            // attach new node w/ value v to n.right
    rebalance(n);
AVL Insertion
AVL Insertion

First: is this an AVL tree?
AVL Insertion

```
insert(Node n, int v):
    // ...(other case, irrelevant here)
    else: // v > n.value
        if n has right:
            insert(n.right, v)
        else:
            // attach new node w/ value
            //   v to n.right
            rebalance(n);
```

```
insert(a, 16)
```
AVL Insertion

```
insert(Node n, int v):
    // ...(other case, irrelevant here)
else:  // v > n.value
    if n has right:
        insert(n.right, v)
    else:
        // attach new node w/ value
        //   v to n.right
    rebalance(n);
```

```
insert(a, 16)
=>insert(c, 16)
```

rebalance(a)
AVL Insertion

```python
insert(Node n, int v):
    //...(other case, irrelevant here)
    else: // v > n.value
        if n has right:
            insert(n.right, v)
        else:
            // attach new node w/ value
            // v to n.right
            rebalance(n);
```

```
insert(a, 16)
=>insert(c, 16)
=>insert(f, 16)
```

rebalance(c)
rebalance(a)
AVL Insertion

\[
\text{insert}(\text{Node } n, \text{ int } v):
\]
// ... (other case, irrelevant here)
else: // v > n.value
  if n has right:
    \text{insert}(n.\text{right}, v)
  else:
    // attach new node w/ value
    //  v to n.\text{right}
\text{rebalance}(n);

insert(a, 16)
=> insert(c, 16)
  => insert(f, 16)
    => attach new node
    // v to n.\text{right}
    \text{rebalance}(f)
    \text{rebalance}(c)
    \text{rebalance}(a)
AVL Insertion

```
insert(Node n, int v):
  // ...(other case, irrelevant here)
  else: // v > n.value
    if n has right:
      insert(n.right, v)
    else:
      // attach new node w/ value
      //   v to n.right
      rebalance(n);
```

\[
\begin{array}{c}
\text{insert}(a, 16) \\
\Rightarrow \text{insert}(c, 16) \\
\Rightarrow \text{insert}(f, 16) \\
\Rightarrow \text{attach new node already balanced} \\
\text{rebalance}(f) \\
\text{rebalance}(c) \\
\text{rebalance}(a)
\end{array}
\]
AVL Insertion

```python
insert(Node n, int v):
    //...(other case, irrelevant here)
    else: // v > n.value
        if n has right:
            insert(n.right, v)
        else:
            // attach new node w/ value
            // v to n.right
            rebalance(n);
```

- `insert(a, 16)`
  => `insert(c, 16)`
  => `insert(f, 16)`
  => `attach new node` `already balanced`
  `rebalance(f)` `perform rotation`
  `rebalance(c)`
  `rebalance(a)`
AVL Insertion

\[
\text{insert}(\text{Node } n, \text{ int } v):
\]
\[
\text{// ...(other case, irrelevant here)}
\]
\[
\text{else: } \text{// } v > n.\text{value}
\]
\[
\text{if } n \text{ has right:}
\]
\[
\text{insert}(n.\text{right}, v)
\]
\[
\text{else:}
\]
\[
\text{// attach new node w/ value}
\]
\[
\text{// } v \text{ to } n.\text{right}
\]
\[
\text{rebalance}(n);
\]

\[
\text{insert}(a, 16)
\]
\[
=> \text{insert}(c, 16)
\]
\[
=> \text{insert}(f, 16)
\]
\[
=> \text{attach new node}
\]
\[
\text{rebalance}(f) \text{ already balanced}
\]
\[
\text{rebalance}(c) \text{ perform rotation}
\]
\[
\text{rebalance}(a)
\]
AVL Insertion

```plaintext
insert(Node n, int v):
    //...(other case, irrelevant here)
    else: // v > n.value
        if n has right:
            insert(n.right, v)
        else:
            // attach new node w/ value
            // v to n.right
            rebalance(n);

insert(a, 16)
=>insert(c, 16)
    =>insert(f, 16)
    =>attach new node
        rebalance(f) already balanced
        rebalance(c) perform rotation
        rebalance(a) already balanced
```
Order of actual execution

```
insert rebalance  
insert rebalance  
insert rebalance  
```

```
10  
8  
-1  
-4  
9  
11  
15  
16  
```
AVL Insertion

insert(Node n, int v):
    //...(other case, irrelevant here)
    else: // v > n.value
        if n has right:
            insert(n.right, v)
        else:
            // attach new node w/ value
            //   v to n.right
            rebalance(n);

insert(a, 16)
=>insert(c, 16)
=>insert(f, 16)
=>attach new node
   rebalance(f)
   rebalance(c)
   rebalance(a) already balanced

How did we know what rotation to do?
Reminder: Tree Rotations

subtrees (could be null, leaf, or tree with many nodes)
AVL Rebalance

Before an insertion that unbalances n, the tree must look like one of these:
AVL Rebalance

Before an insertion that unbalances n, the tree must look like one of these:

An insertion that unbalances n could go one of four places.
AVL Rebalance

Before an insertion that unbalances $n$, the tree must look like one of these:

Case 1

Case 2

An insertion that unbalances $n$ could go one of four places.
AVL Rebalance

Case 1: After BST insertion step, the tree looks like this.
AVL Rebalance

Case 1: After BST insertion step, the tree looks like this.
AVL Rebalance

Case 1: After BST insertion step, the tree looks like this.
Solution: right rotate on N.
AVL Rebalance

**Case 1:** After BST insertion step, the tree looks like this.

Solution: right rotate on N.
AVL Rebalance

Case 1: After BST insertion step, the tree looks like this.
Solution: right rotate on N.
N is now AVL balanced.
Before an insertion that unbalances $n$, the tree must look like one of these:

Case 1

Case 2

Case 3

Case 4

An insertion that unbalances $n$ could gone one of four places.
AVL Rebalance

Case 2: After BST insertion step, the tree looks like this.
Case 2: After BST insertion step, the tree looks like this.

![AVL Tree Diagram]

**AVL Rebalance**

- **Case 2**: After BST insertion step, the tree looks like this.
AVL Rebalance

Case 2: After BST insertion step, the tree looks like this.

Solution - two rotations:
1. Left rotate C
2. Right rotate N
AVL Rebalance

Case 2: After BST insertion step, the tree looks like this.

Solution - two rotations:
1. Left rotate C
2. Right rotate N
AVL Rebalance

Case 2: After BST insertion step, the tree looks like this.

Solution - two rotations:
1. Left rotate C
2. Right rotate N
AVL Rebalance

Case 2: After BST insertion step, the tree looks like this.

Solution - two rotations:
1. **Left rotate C**
2. **Right rotate N**
AVL Rebalance

Case 2: After BST insertion step, the tree looks like this.

Solution - two rotations:
1. Left rotate C
2. Right rotate N
AVL Rebalance

Case 2: After BST insertion step, the tree looks like this.

Solution - two rotations:
1. Left rotate C
2. Right rotate N
AVL Rebalance

Case 2: After BST insertion step, the tree looks like this.

Solution - two rotations:
1. Left rotate C
2. Right rotate N

Tree is now AVL balanced.
AVL Rebalance

Before an insertion that unbalances n, the tree must look like one of these:

An insertion that unbalances n could have gone one of four places.
void rebalance(n):
  if bal(n) < -1:
    if bal(n.left) < 0
      // case 1:
      // rightRot(n)
    else:
      // case 2:
      // leftRot(n.L);
      // rightRot(n)
  else if bal(n) > 1:
    if bal(n.right) < 0:
      // case 3:
      // rightRot(n.R);
      // leftRot(n)
    else:
      // case 4:
      // leftRot(n)
void rebalance(n):
    if bal(n) < -1:
        if bal(n.left) < 0
            // case 1:
            // rightRot(n)
        else:
            // case 2:
            // leftRot(n.L);
            // rightRot(n)
    else if bal(n) > 1:
        if bal(n.right) < 0:
            // case 3:
            // rightRot(n.R);
            // leftRot(n)
        else:
            // case 4:
            // leftRot(n)
void rebalance(n):
    if bal(n) < -1:
        if bal(n.left) < 0
            // case 1:
            // rightRot(n)
        else:
            // case 2:
            // leftRot(n.L);
            // rightRot(n)
    else if bal(n) > 1:
        if bal(n.right) < 0:
            // case 3:
            // rightRot(n.R);
            // leftRot(n)
        else:
            // case 4:
            // leftRot(n)
void rebalance(n):
    if bal(n) < -1:
        if bal(n.left) < 0
            // case 1:
            // rightRot(n)
        else:
            // case 2:
            // leftRot(n.L);
            // rightRot(n)
    else if bal(n) > 1:
        if bal(n.right) < 0:
            // case 3:
            // rightRot(n.R);
            // leftRot(n)
        else:
            // case 4:
            // leftRot(n)

Cases 3 and 4 are symmetric with 2 and 1
Implementation

```cpp
void rebalance(n):
    if bal(n) < -1:
        if bal(n.left) < 0
            // case 1:
            // rightRot(n)
        else:
            // case 2:
            // leftRot(n.L);
            // rightRot(n)
    else if bal(n) > 1:
        if bal(n.right) < 0:
            // case 3:
            // rightRot(n.R);
            // leftRot(n)
        else:
            // case 4:
            // leftRot(n)
```

Cases 3 and 4 are symmetric with 2 and 1.
void rebalance(n):
    if bal(n) < -1:
        if bal(n.left) < 0
            // case 1:
            // rightRot(n)
    else:
        // case 2:
        // leftRot(n.L);
        // rightRot(n)
    else if bal(n) > 1:
        if bal(n.right) < 0:
            // case 3:
            // rightRot(n.R);
            // leftRot(n)
        else:
            // case 4:
            // leftRot(n)

Details

• Implementing bal:
• Nodes track their height
• Need to update when the tree changes
• Edge cases near the root

Cases 3 and 4 are symmetric with 2 and 1.
Insertation with Rebalance

How did we know what rotation to do?

insert(Node n, int v):
    //...(other case, irrelevant here)
    else:  // v > n.value
        if n has right:
            insert(n.right, v)
        else:
            // attach new node w/ value
            //   v to n.right
            rebalance(n);

insert(a, 16)
=>insert(c, 16)
=>insert(f, 16)
=>attach new node
   already balanced
   rebalance(f)
   perform rotation
rebalance(c)
already balanced
rebalance(a)
Height of AVL Trees

- As usual, runtime of search, insert, and remove are all $O(\text{height})$.

- A rotation is $O(1)$, so even if we have to rebalance every node on the path to the root, it’s still only $h \times O(1)$ rebalances.
Height of AVL Trees

• As usual, runtime of search, insert, and remove are all $O(\text{height})$

• How many nodes in an AVL tree of height $h$?

• or, what’s the tallest tree you can get with $n$ nodes?
  • Exact proof involves fibonacci sequence(!)

• To add to root’s height, you have to add to height of every subtree in one of one of root’s subtrees.
Removing from AVL Tree

• Much like insertion: remove as usual, rebalance as necessary at each level up to the root.

• Whereas insertion only ever requires only one rebalance, deletion can require many
  
  • but still no more than the tree’s height.