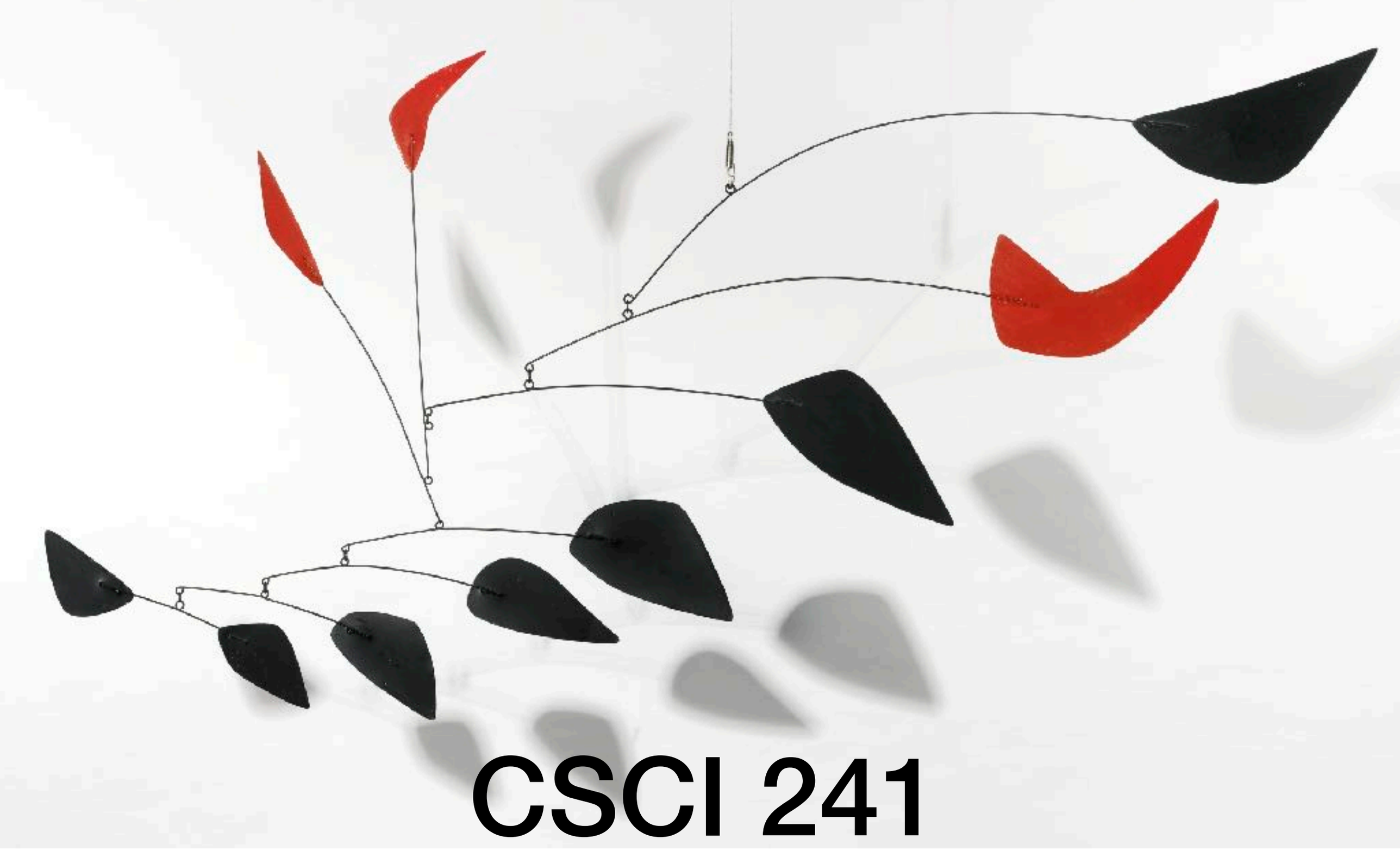


CSCI 241

Lecture 12

AVL Trees, Continued

Happenings

Happenings this week:

- Tuesday, Oct. 23 – Peer Lecture Series: Command Line Basics – 4 pm in CF 420
- Wednesday, Oct. 24 – Tech Talk: Accenture – 5 pm in CF 125
- Thursday, Oct. 25 – Group Advising to Declare the CS Major! – 3 pm in CF 420
- Friday, Oct. 26 – Cybersecurity Internship Workshop – 12 pm in AW 203

Goals

- Be prepared to implement AVL rebalancing.

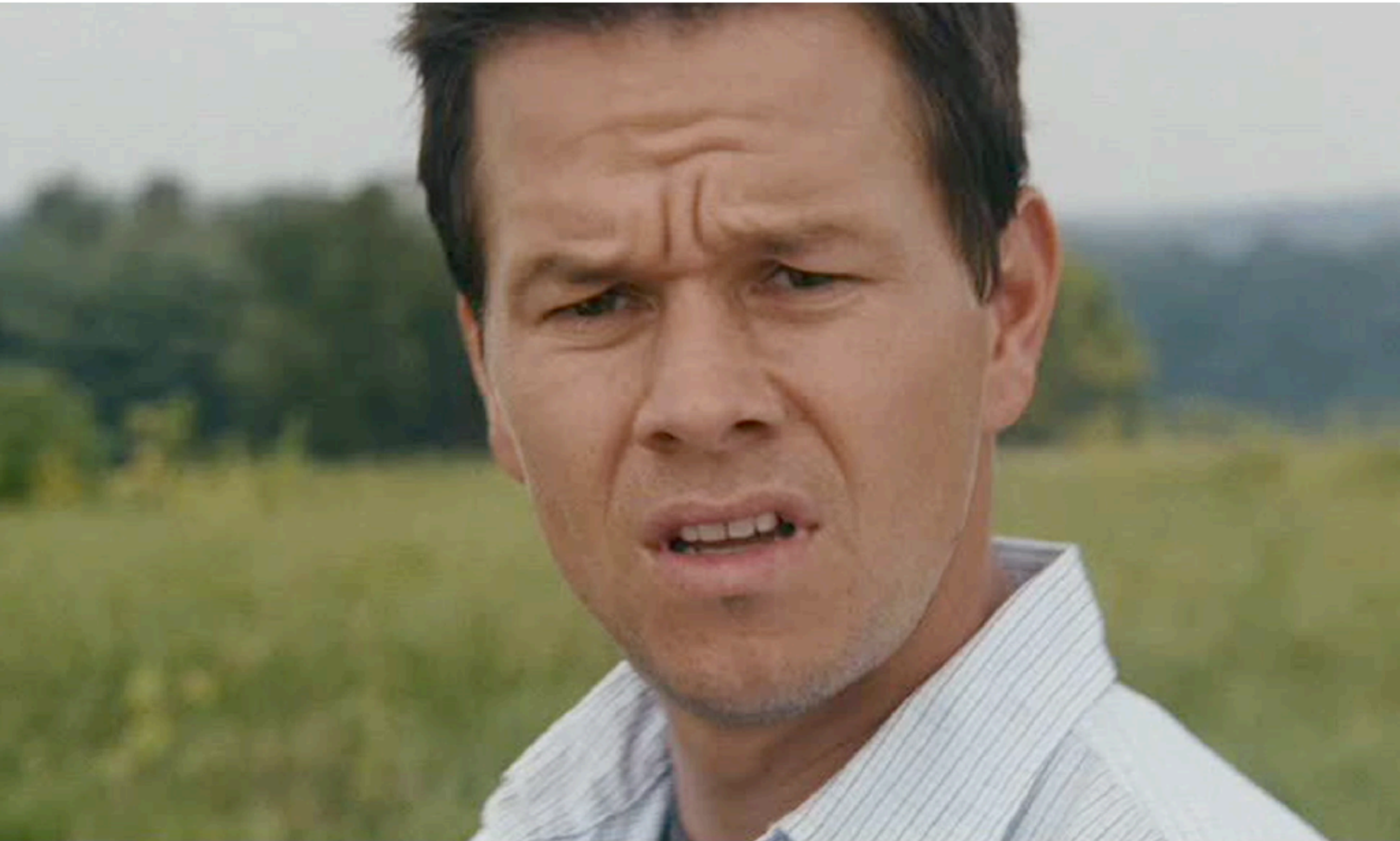
Announcements

- Quiz 3 is not graded yet =(
- A1 will be out tonight. It will be due Monday 10/29.
- How to see your Quiz 2:

  1. Go to www.gradescope.com
  2. Click Log In (*not* Sign Up - your account exists already)
  3. Click “Forgot your password?”, enter your WWU email address.
  4. Set your password using the link you receive by email.
  5. Log into gradescope.com using this password.

Where we left off on Friday

Where we left off on Friday



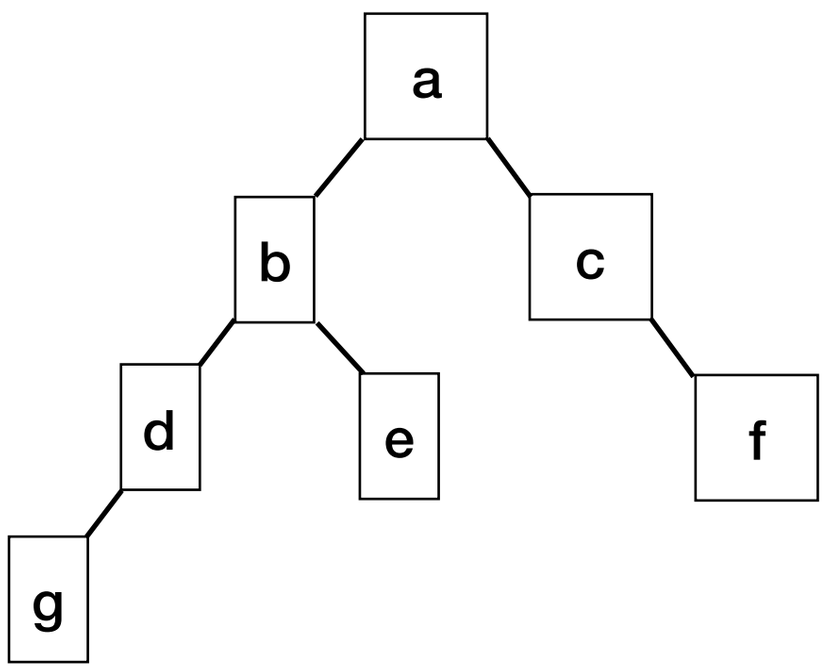
What are we even doing

- Trees
- Binary Trees
- Binary Search Trees
- Balanced BSTs
- AVL trees

What are we even doing

- Trees what are they? nodes with 0 or more children (subtrees)
- Binary Trees nodes with 0,1, or 2 children (subtrees)
- Binary Search Trees | **search** | **insert** | **remove** | and their **runtimes**
- Balanced BSTs why do we want this?
  balanced trees give good runtime
- AVL trees how do we measure balance?
  balance factor
  how do we achieve balance?
  rotations
  how do we know what/when to rotate?

Heights and Balance Factors



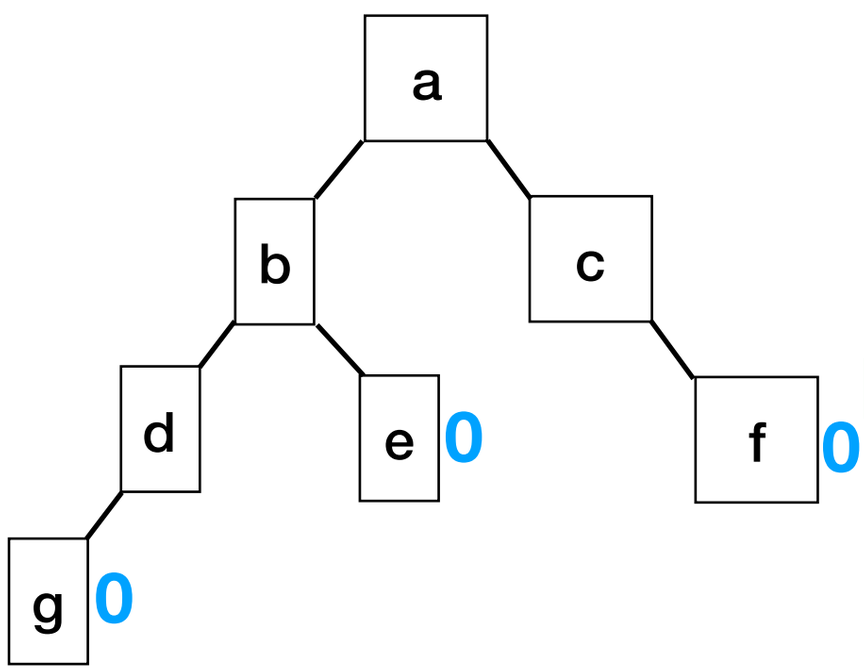
Height(t): path length from t's deepest descendant (leaf) to t's root.

Height(n): height of the subtree rooted at n

Balance(n): height(n.right) - height(n.left)

```
height(null) = -1
height(n) = 1 + max(height(n.left), height(n.right)
```

Heights and Balance Factors



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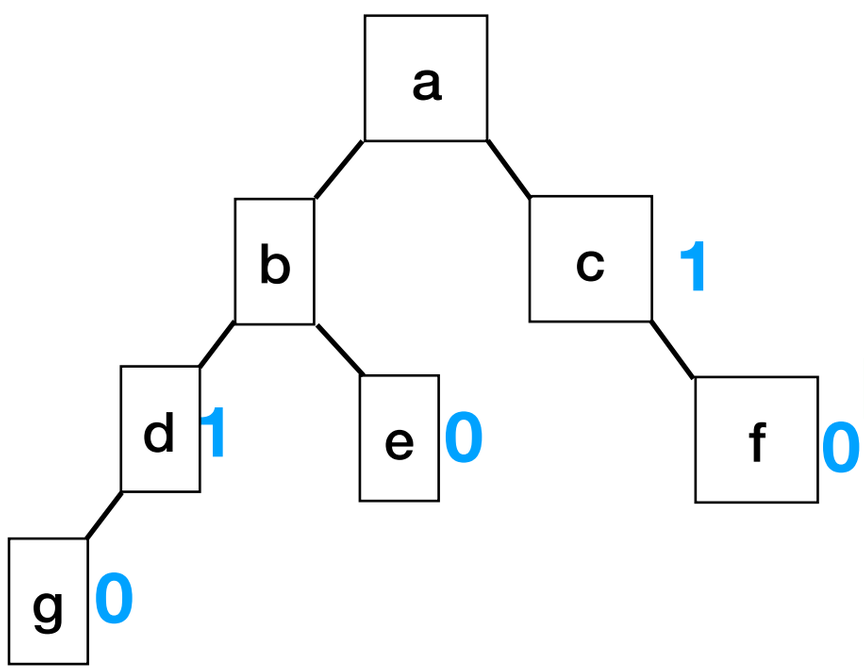

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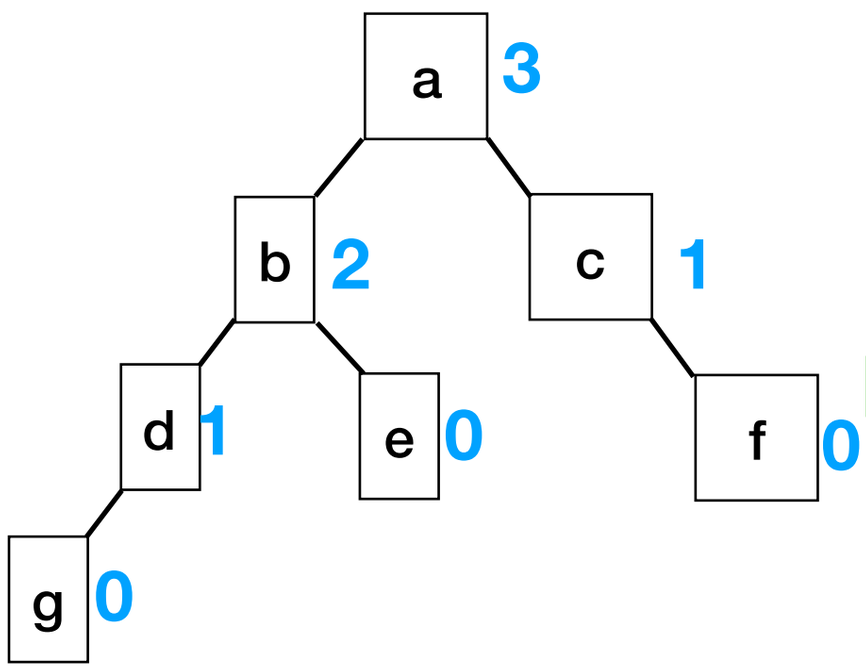
Heights and Balance Factors



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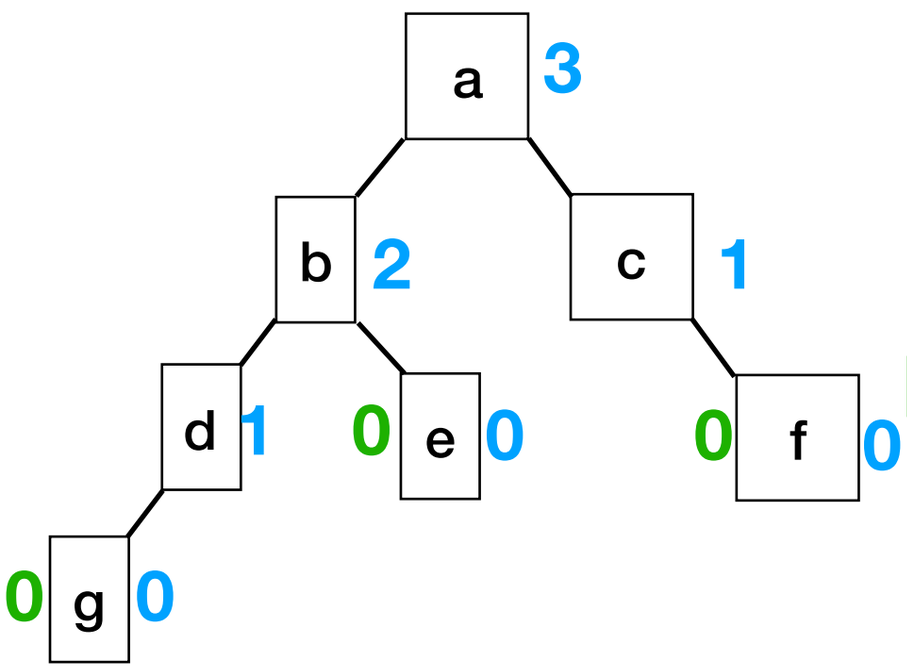

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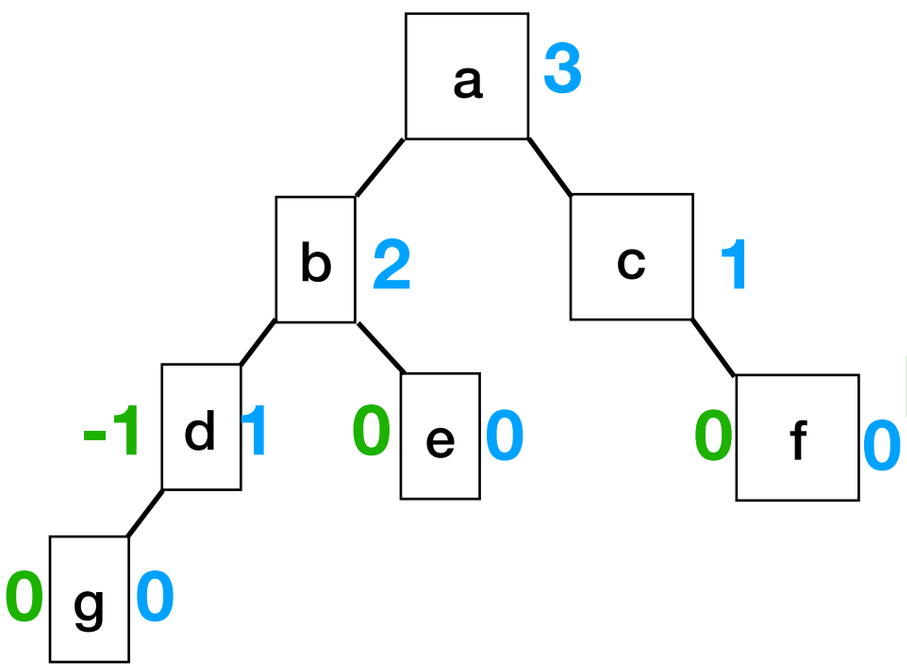
Heights and Balance Factors



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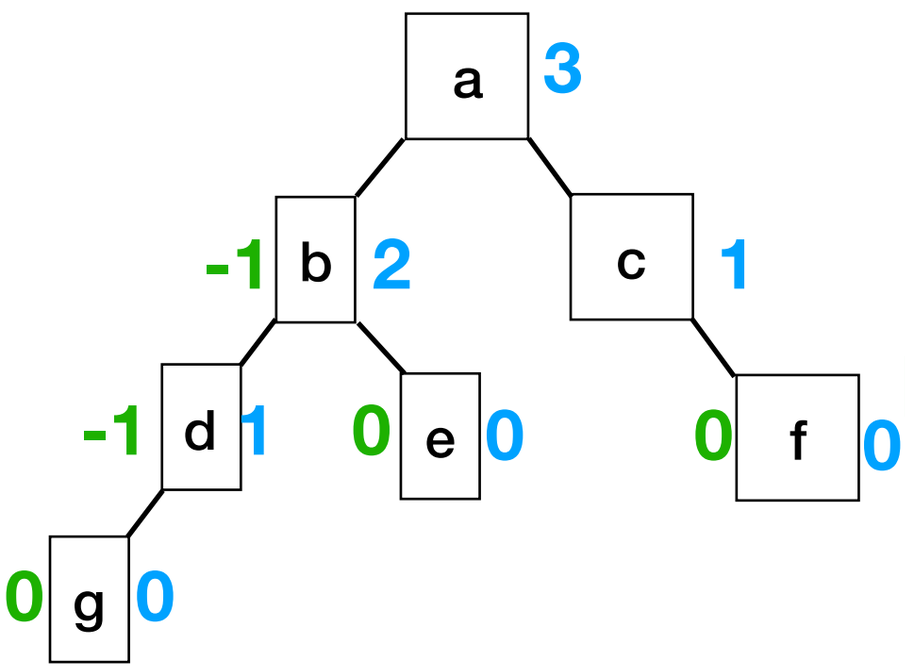
Heights and Balance Factors



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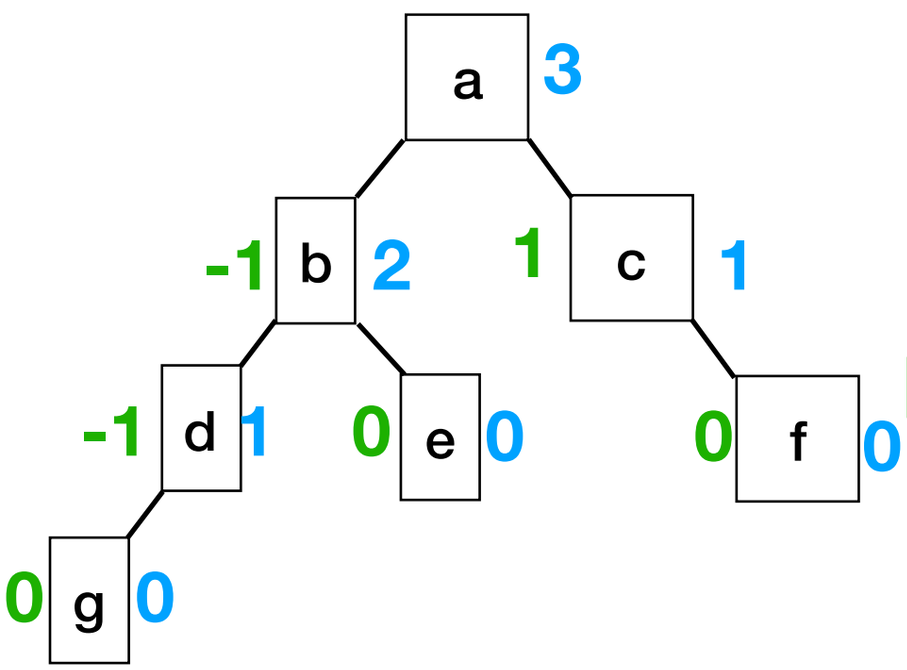
Heights and Balance Factors



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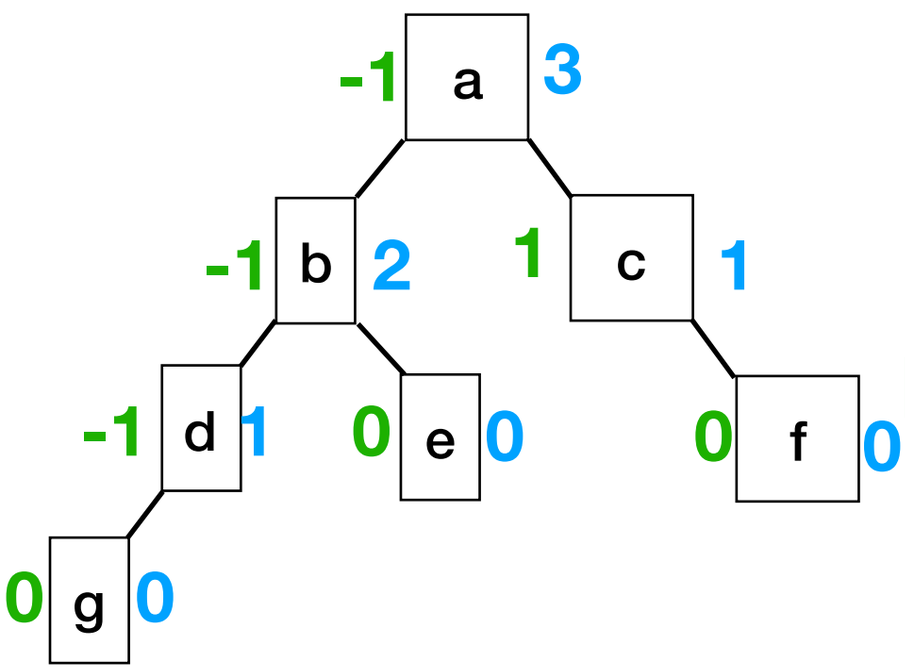
Heights and Balance Factors



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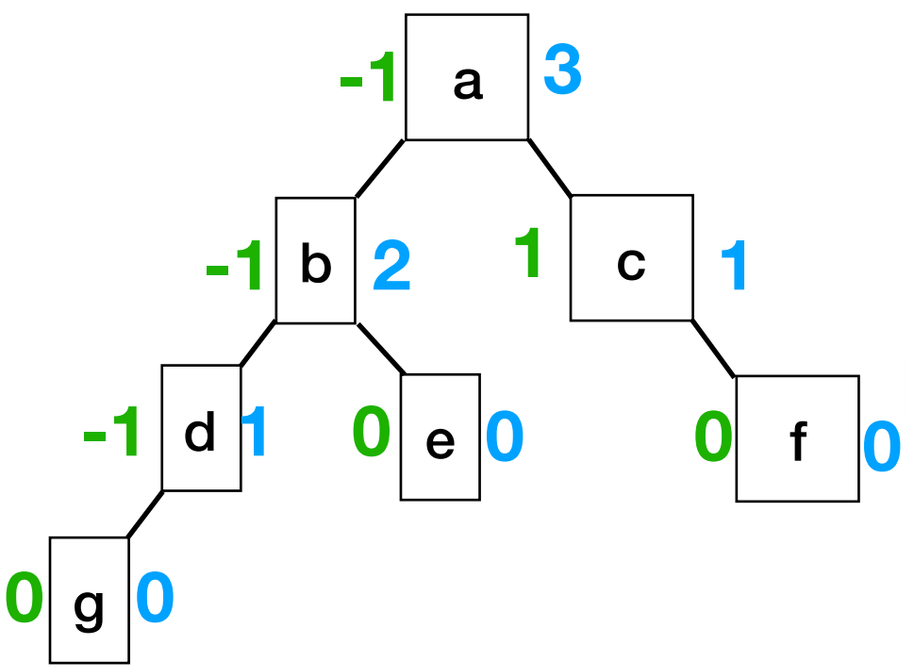
Heights and Balance Factors



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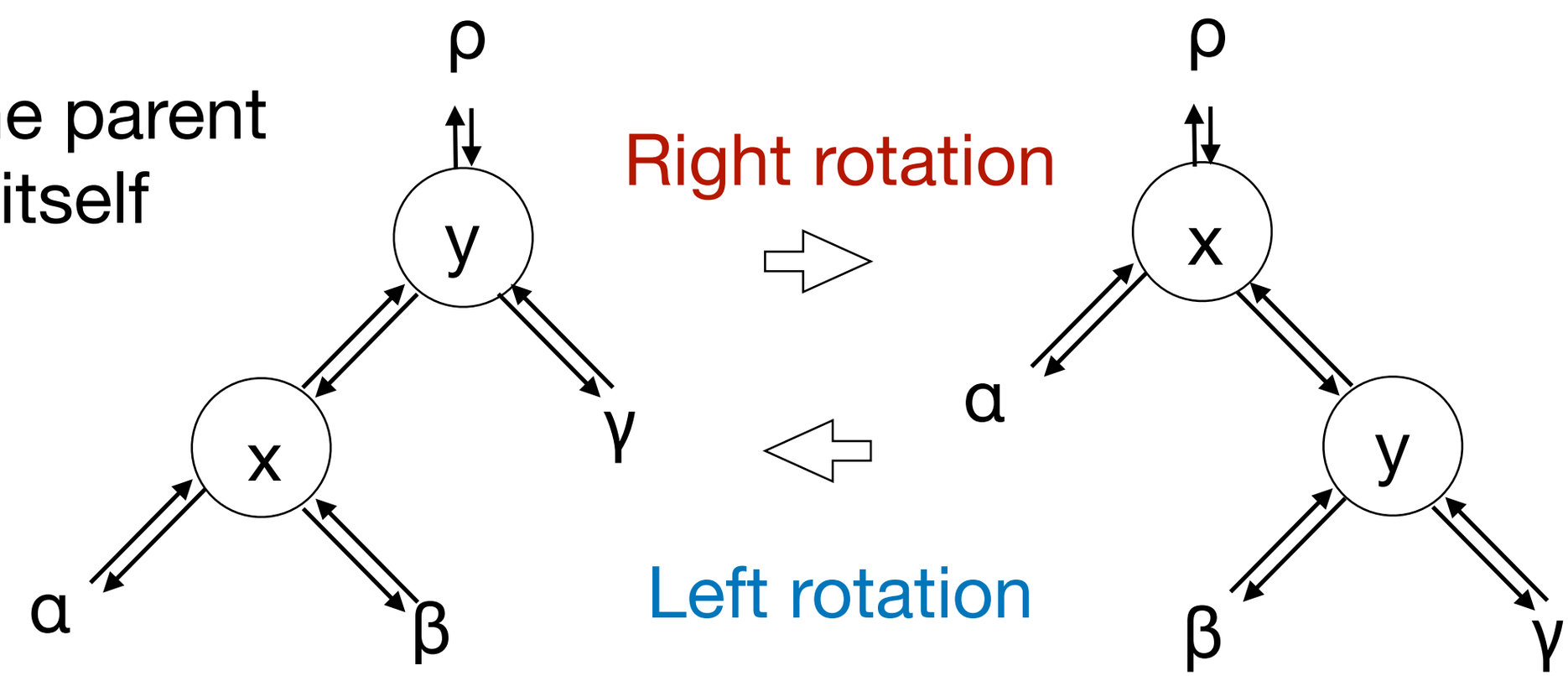
```

Tree Rotations

Steps in left rotation (move y up to x's position):

1. Transfer β
2. Transfer the parent
3. Transfer x itself



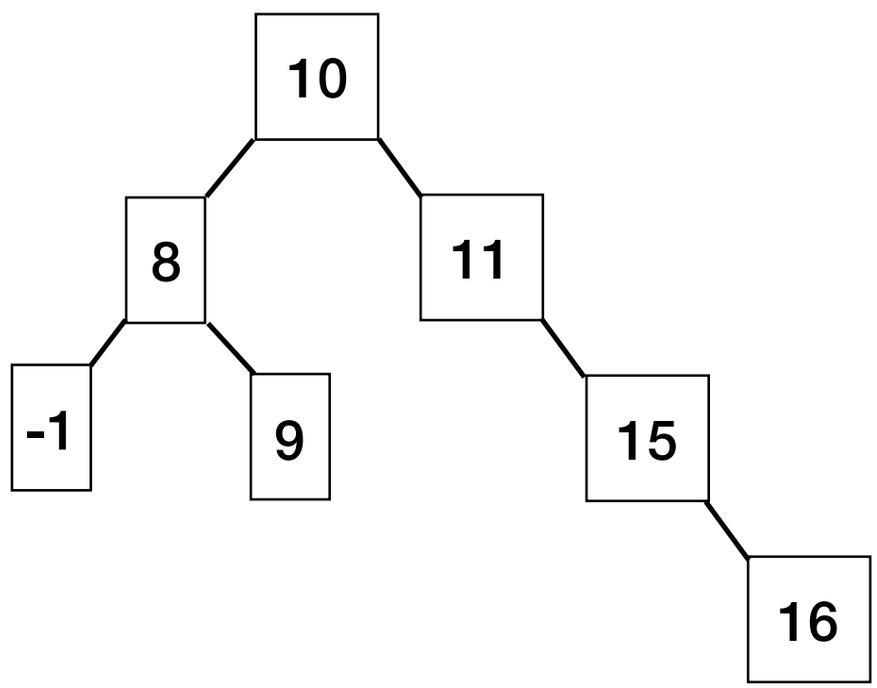
x.R gets y.L
y.L.p gets x

y.p gets x.p
p.[L/R] gets y

y.L gets x
x.p gets y

Can we improve balance?

Balance(n): height(n.right) - height(n.left)



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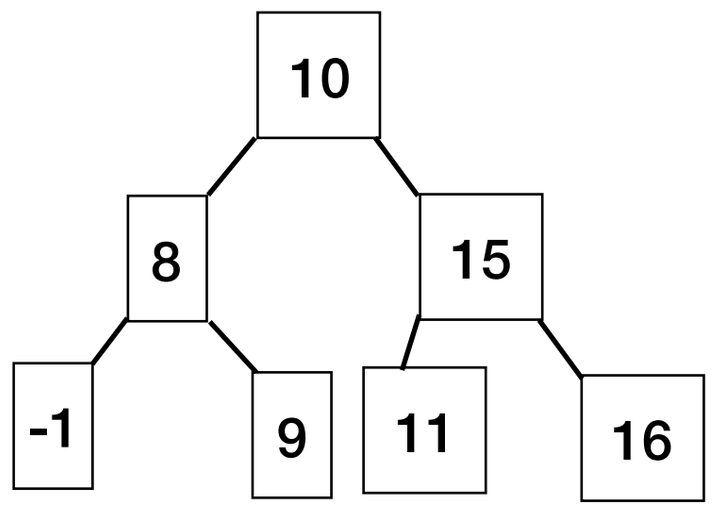

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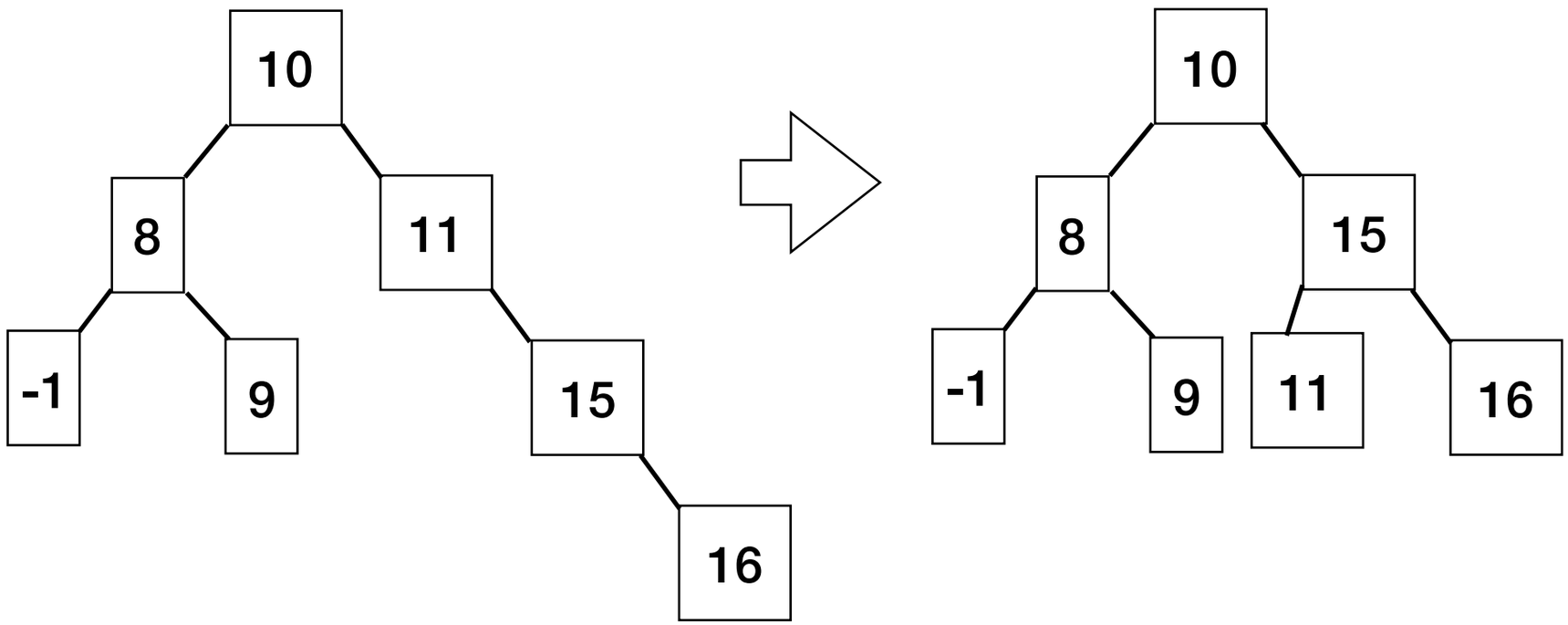
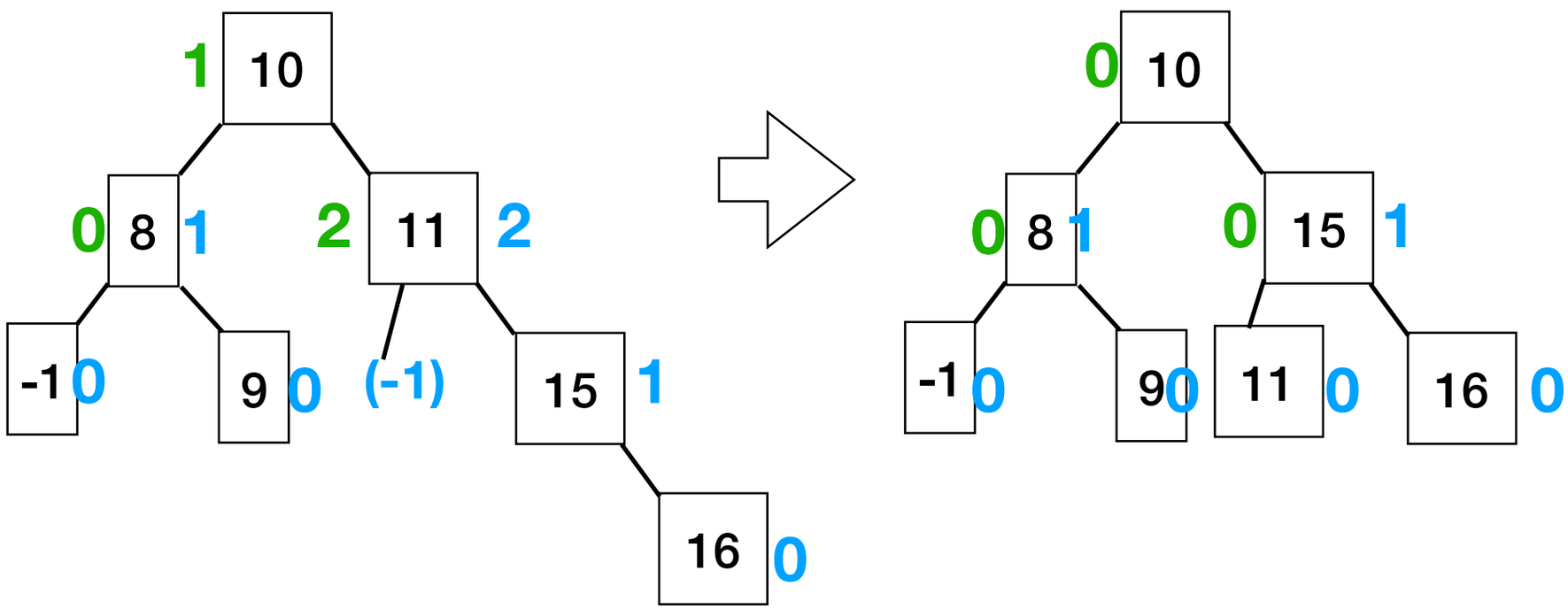


ABCD: The largest balance factor in the tree is:

A. 0 before, 1 after rotation
B. 1 before, 0 after rotation
C. 0 before, 2 after rotation
D. 2 before, 1 after rotation

Can we improve balance?

Balance(n): height(n.right) - height(n.left)

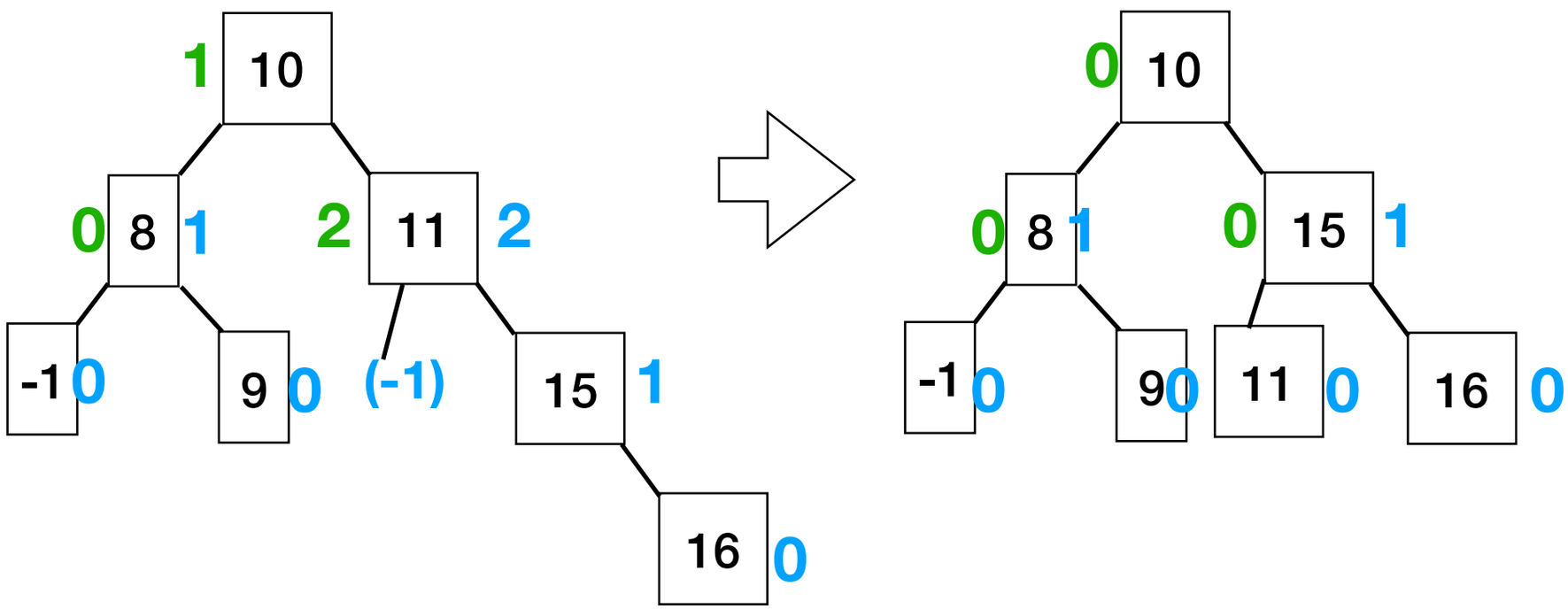


ABCD: The largest (absolute) balance factor in the tree is:

A. 0 before, 1 after rotation
B. 1 before, 0 after rotation
C. 0 before, 2 after rotation
D. 2 before, 0 after rotation

Can we improve balance?

Balance(n): height(n.right) - height(n.left)



How do we know what to rotate and when?
If the tree changes, check for imbalance and fix it if found.

AVL Trees

Balance(n): height(n.right) - height(n.left)

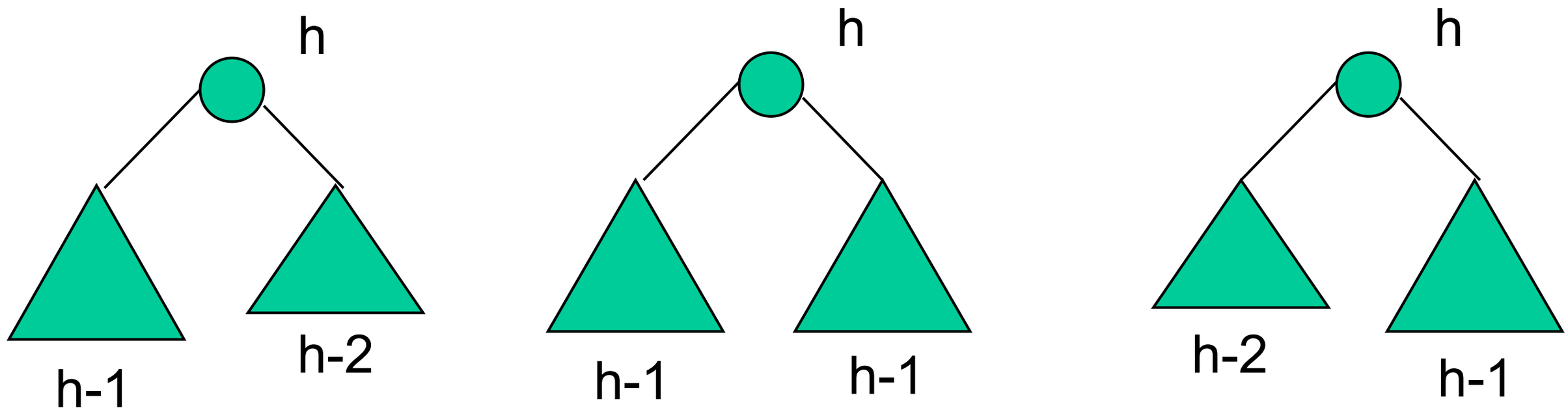
- Devised by **A**delson-**V**elsky and **L**andis
- An AVL tree is a Binary Search Tree in which the following property holds:

AVL property: -1 <= balance(n) <= 1 for all nodes n.

Balance Factor in AVL Trees

AVL property: -1 <= balance(n) <= 1 for all nodes n.

Every subtree in an AVL tree looks like one of these three trees:



(a) Balance factor: 1 (b) Balance factor: 0 (c) Balance factor: -1

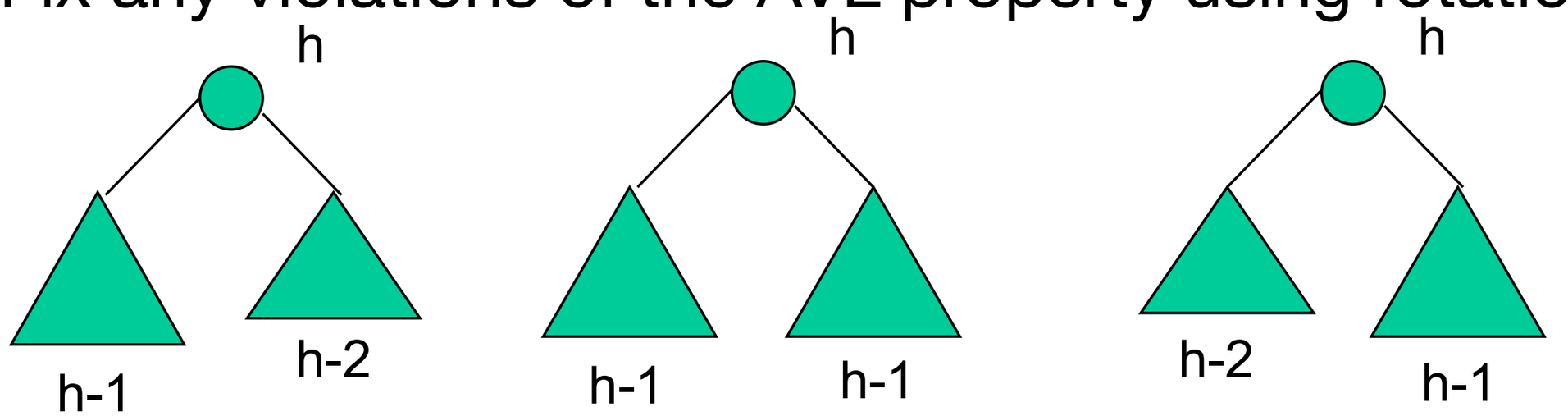
AVL Trees: Insertion

- An AVL tree is a Binary Search Tree in which the following property holds:

AVL property: -1 <= b(n) <= 1 for all nodes n.

To insert into an AVL tree:

1. Do a normal BST insertion
2. Fix any violations of the AVL property using rotations.



(a) Balance factor: 1 (b) Balance factor: 0 (c) Balance factor: -1

AVL Trees: Insertion

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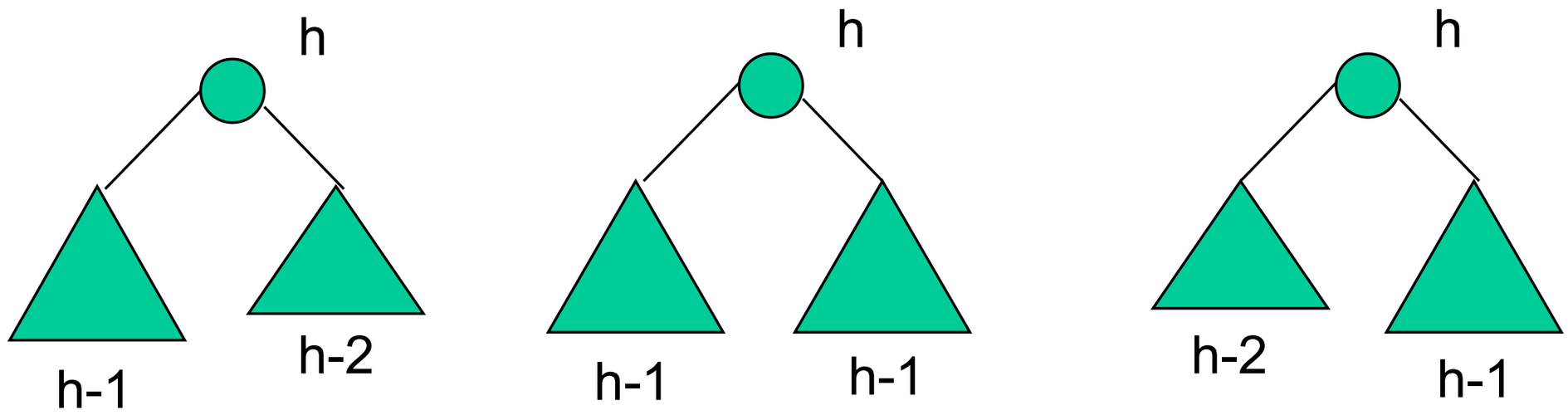

(a) Balance factor: 1 (b) Balance factor: 0 (c) Balance factor: -1

Refresher: BST Insertion

```
/* insert a node with value v into the
 * tree rooted at n. pre: n is not null. */
insert(Node n, int v):
  if n.value == v: return // (duplicate)
  if v < n.value:
    if n has left:
      insert(n.left, v)
    else:
      // attach new node w/ value v to n.left
  else: // v > n.value
    if n has right:
      insert(n.right, v)
    else:
      // attach new node w/ value v to n.right
```

AVL Insertion

```
/* insert a node with value v into the
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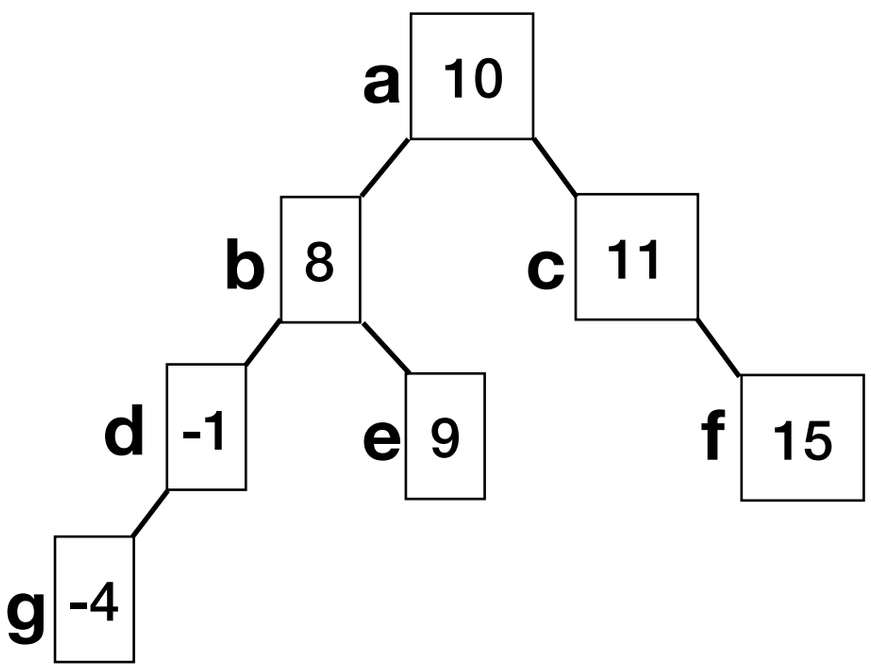
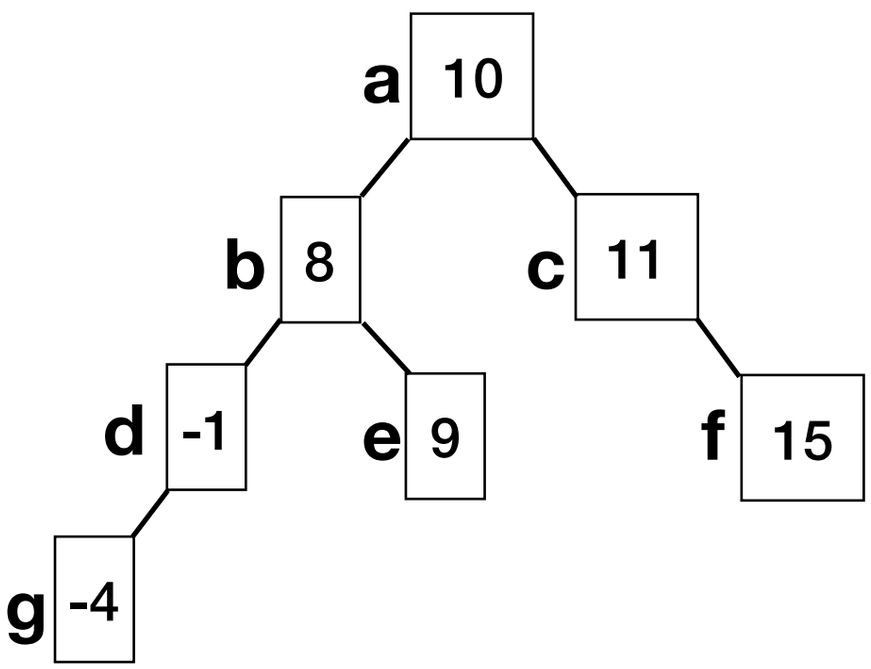
  else: // v > n.value
    if n has right:
      insert(n.right, v)
    else:
      // attach new node w/ value v to n.right
  rebalance(n);
```

AVL Insertion



AVL Insertion

First: is this an AVL tree?



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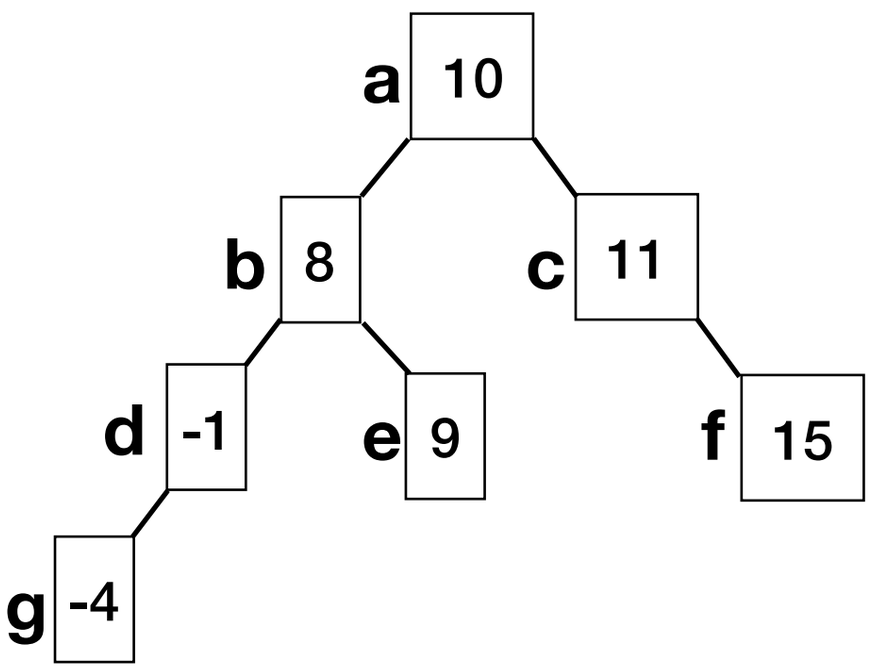
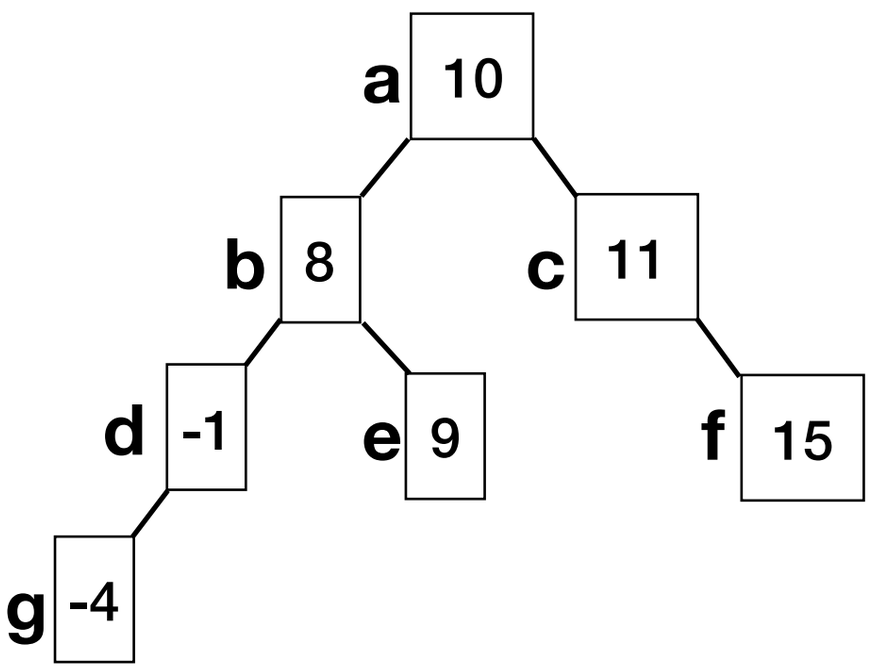



```
insert(Node n, int v):
  //…(other case, irrelevant here)
  else: // v > n.value
    if n has right:
      insert(n.right, v)
    else:
      // attach new node w/ value
      //   v to n.right
rebalance(n);
```

```
insert(a, 16)
```

AVL Insertion

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   else:
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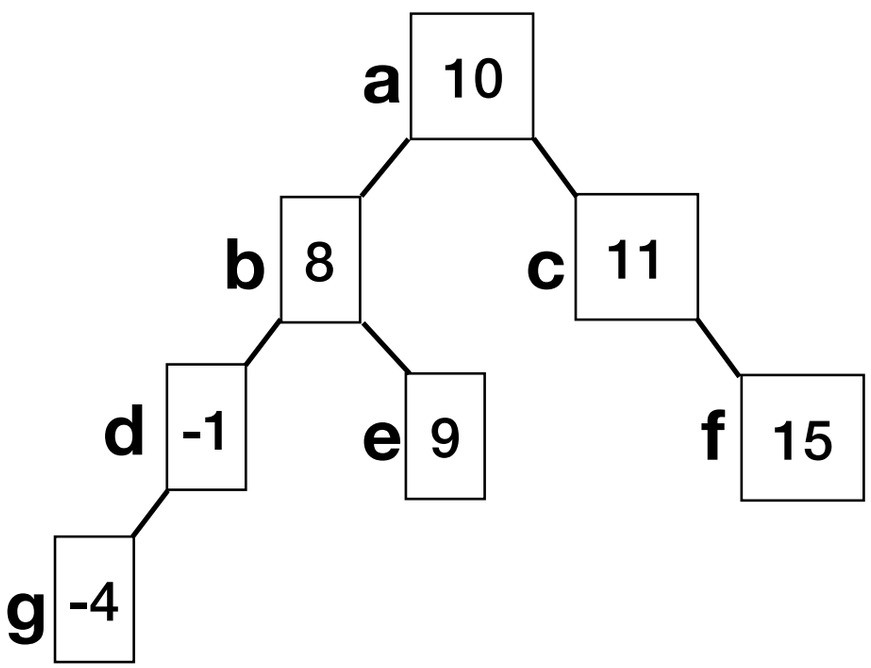
      //   v to n.right
rebalance(n);
```

```
insert(a, 16)
=>insert(c, 16)




  rebalance(a)
```

AVL Insertion

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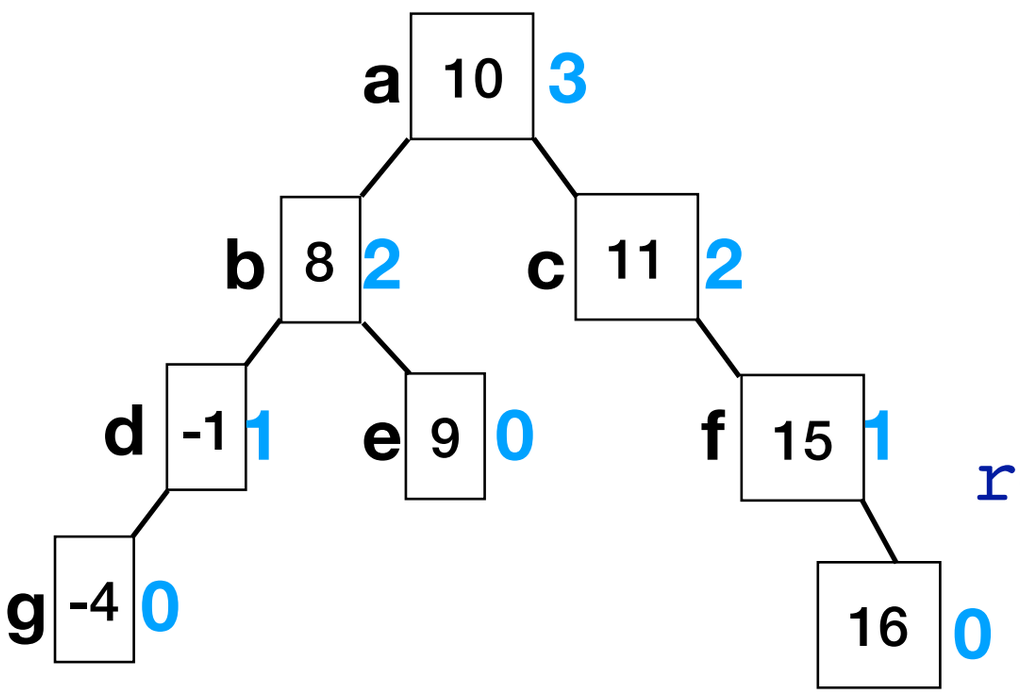
      //   v to n.right
rebalance(n);
```

```
insert(a, 16)
=>insert(c, 16)
  =>insert(f, 16)



    rebalance(c)
  rebalance(a)
```

AVL Insertion

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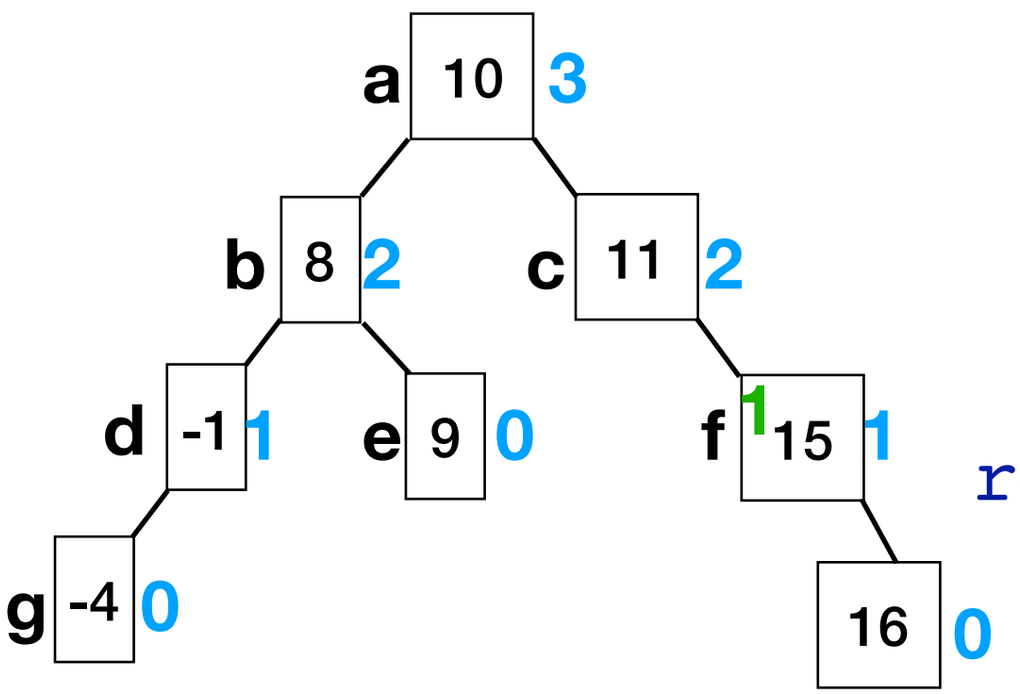
      //    v to n.right
rebalance(n);
```

```
insert(a, 16)
=>insert(c, 16)
  =>insert(f, 16)
    =>attach new node
      rebalance(f)
    rebalance(c)
  rebalance(a)
```

AVL Insertion

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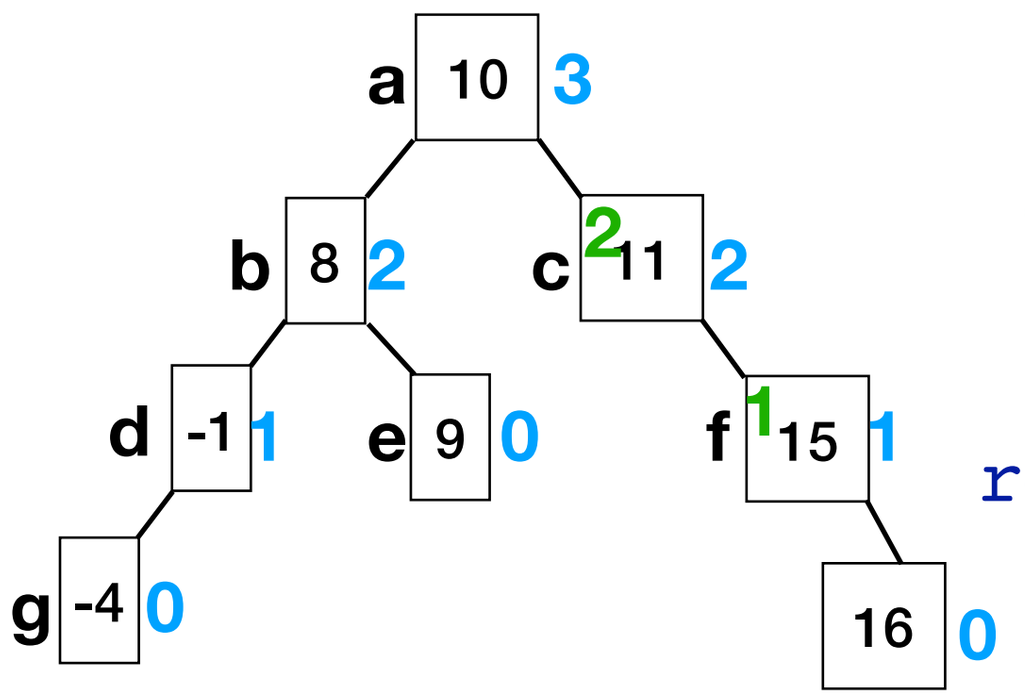
      //   v to n.right
rebalance(n);
```

```
insert(a, 16)
=>insert(c, 16)
  =>insert(f, 16)
    =>attach new node
      rebalance(f)
    rebalance(c)
  rebalance(a)
```

already balanced

AVL Insertion

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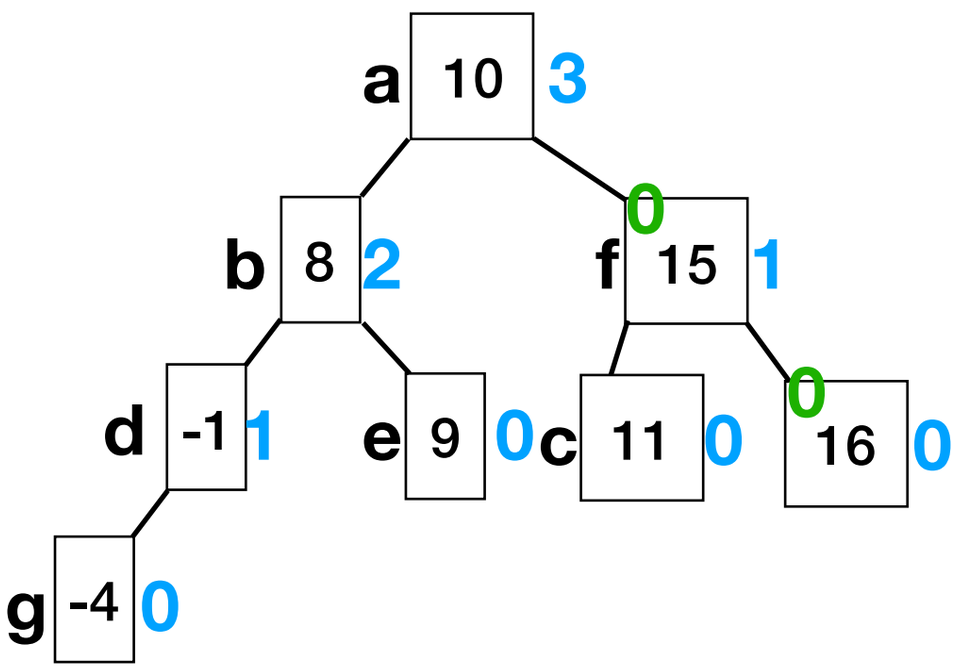
    if n has right:
      insert(n.right, v)
    else:
      // attach new node w/ value
      //   v to n.right
rebalance(n);
```

```
insert(a, 16)
=>insert(c, 16)
  =>insert(f, 16)
    =>attach new node
      rebalance(f)  already balanced
    rebalance(c)  perform rotation
  rebalance(a)
```

AVL Insertion

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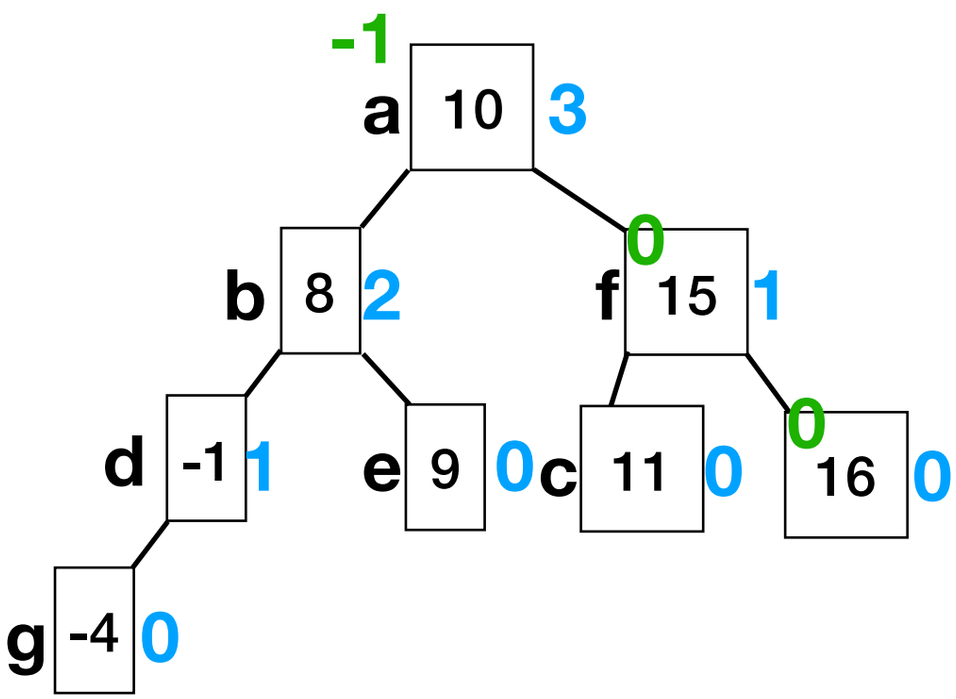
      //   v to n.right
rebalance(n);
```

```
insert(a, 16)
=>insert(c, 16)
  =>insert(f, 16)
    =>attach new node
      rebalance(f) already balanced
    rebalance(c) perform rotation
  rebalance(a)
```

AVL Insertion

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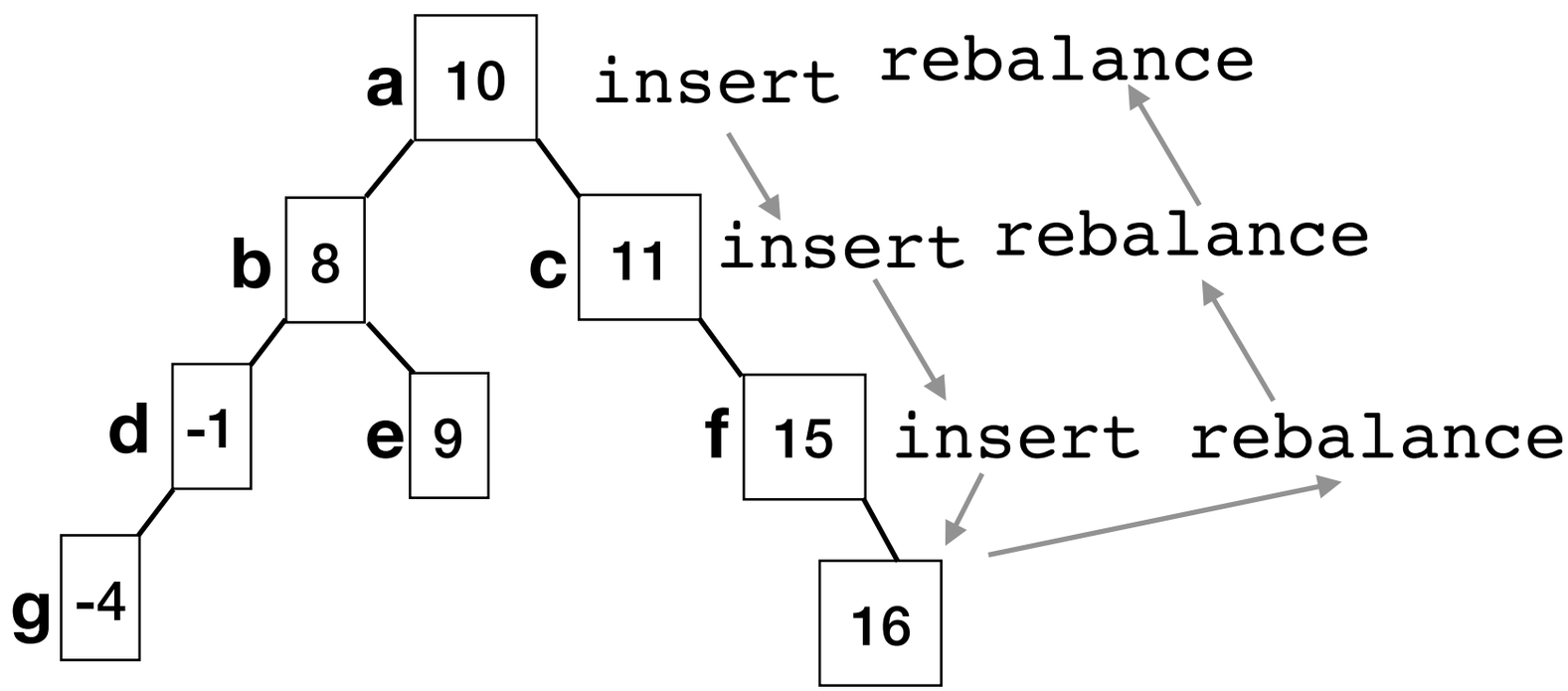
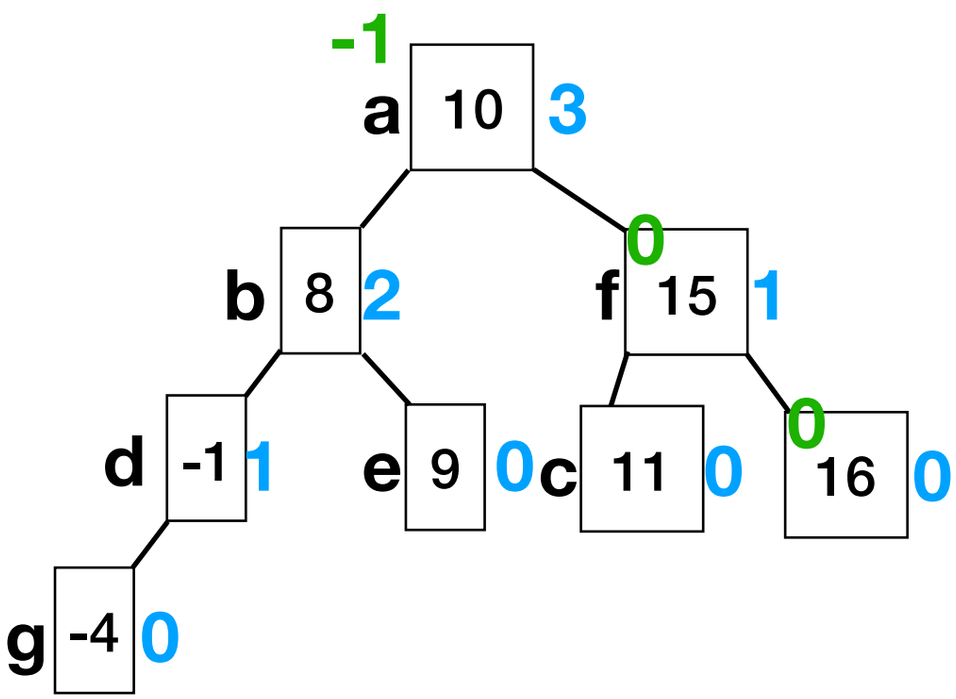
rebalance(n);
```

```
insert(a, 16)
=>insert(c, 16)
  =>insert(f, 16)
    =>attach new node
      rebalance(f) already balanced
    rebalance(c) perform rotation
  rebalance(a) already balanced
```

Order of actual execution



AVL Insertion

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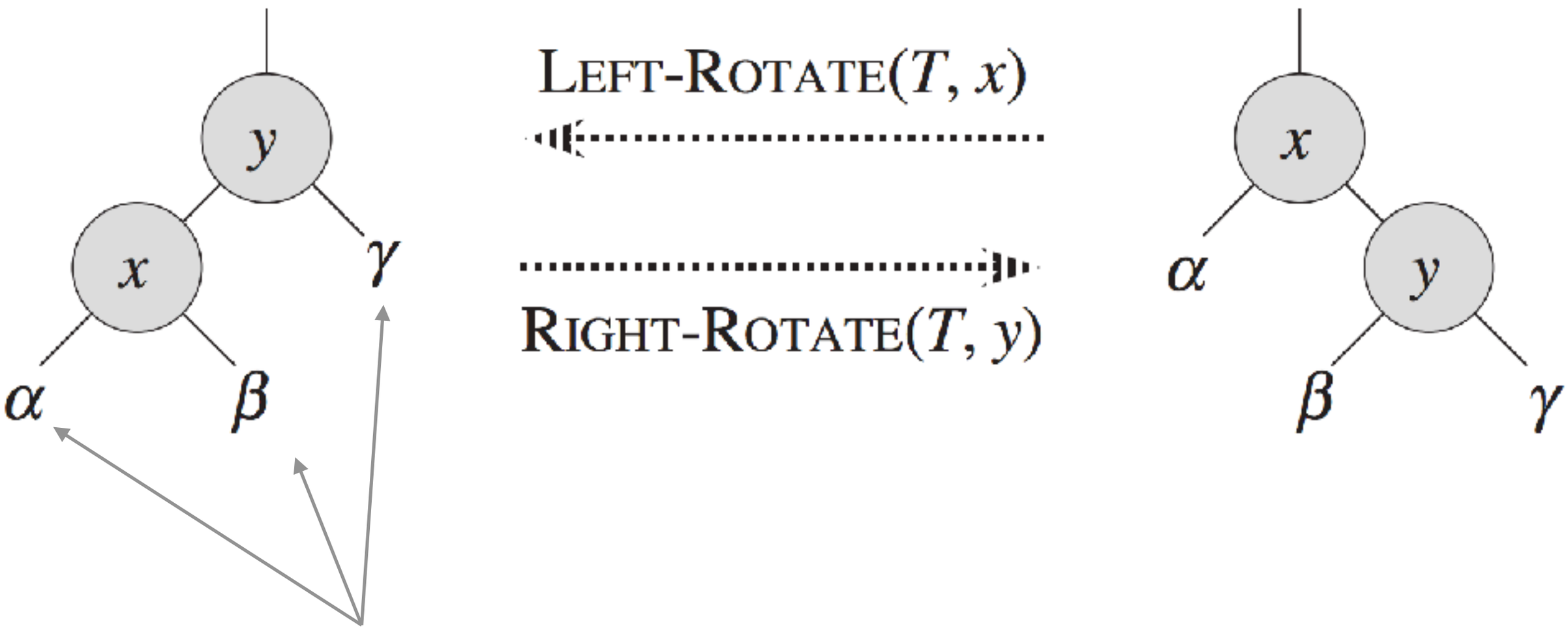
  else: // v > n.value
    if n has right:
      insert(n.right, v)
   else:
      // attach new node w/ value
      //   v to n.right
rebalance(n);
```

How did we know what rotation to do?

```
insert(a, 16)
=>insert(c, 16)
  =>insert(f, 16)
    =>attach new node
      rebalance(f) already balanced
    rebalance(c) perform rotation
  rebalance(a) already balanced
```

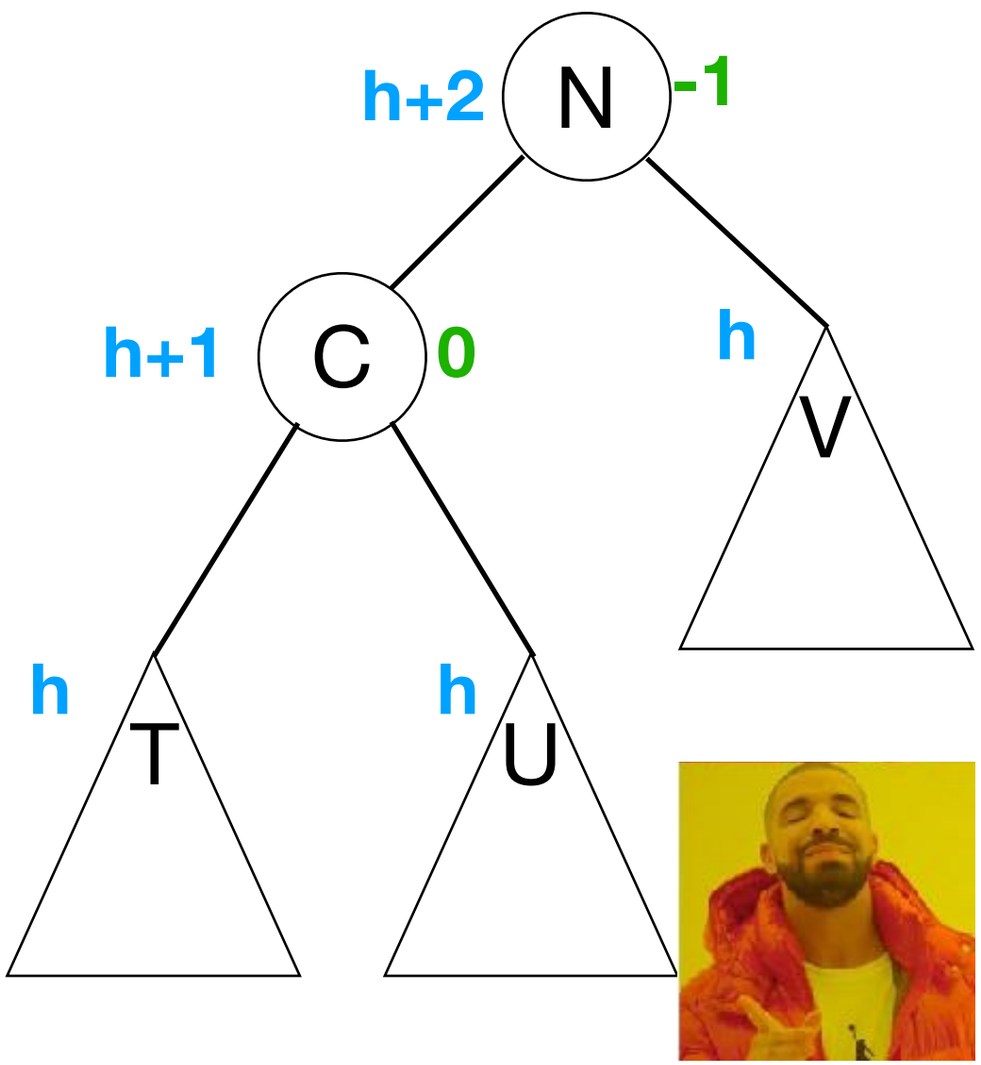
Reminder: Tree Rotations

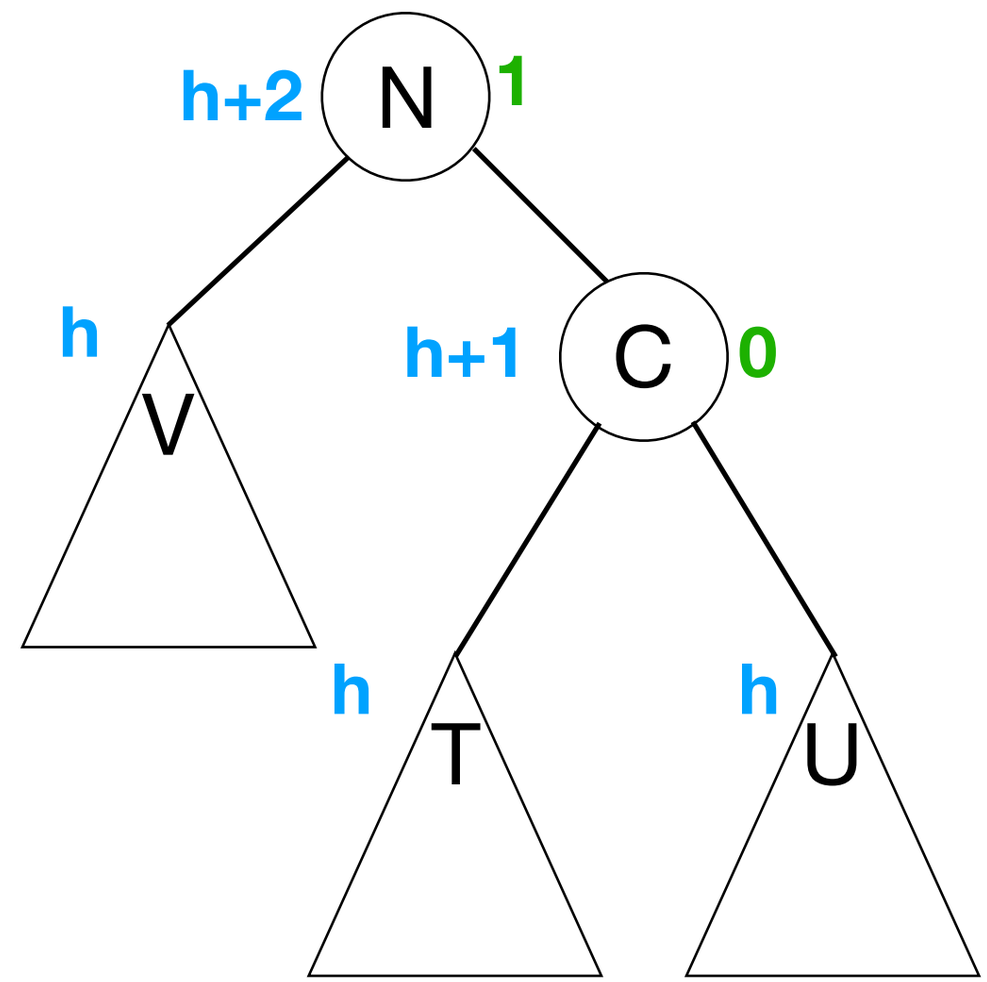


subtrees (could be null, leaf, or tree with many nodes)

AVL Rebalance

Before an insertion that unbalances n,
the tree must look like one of these:





AVL Rebalance

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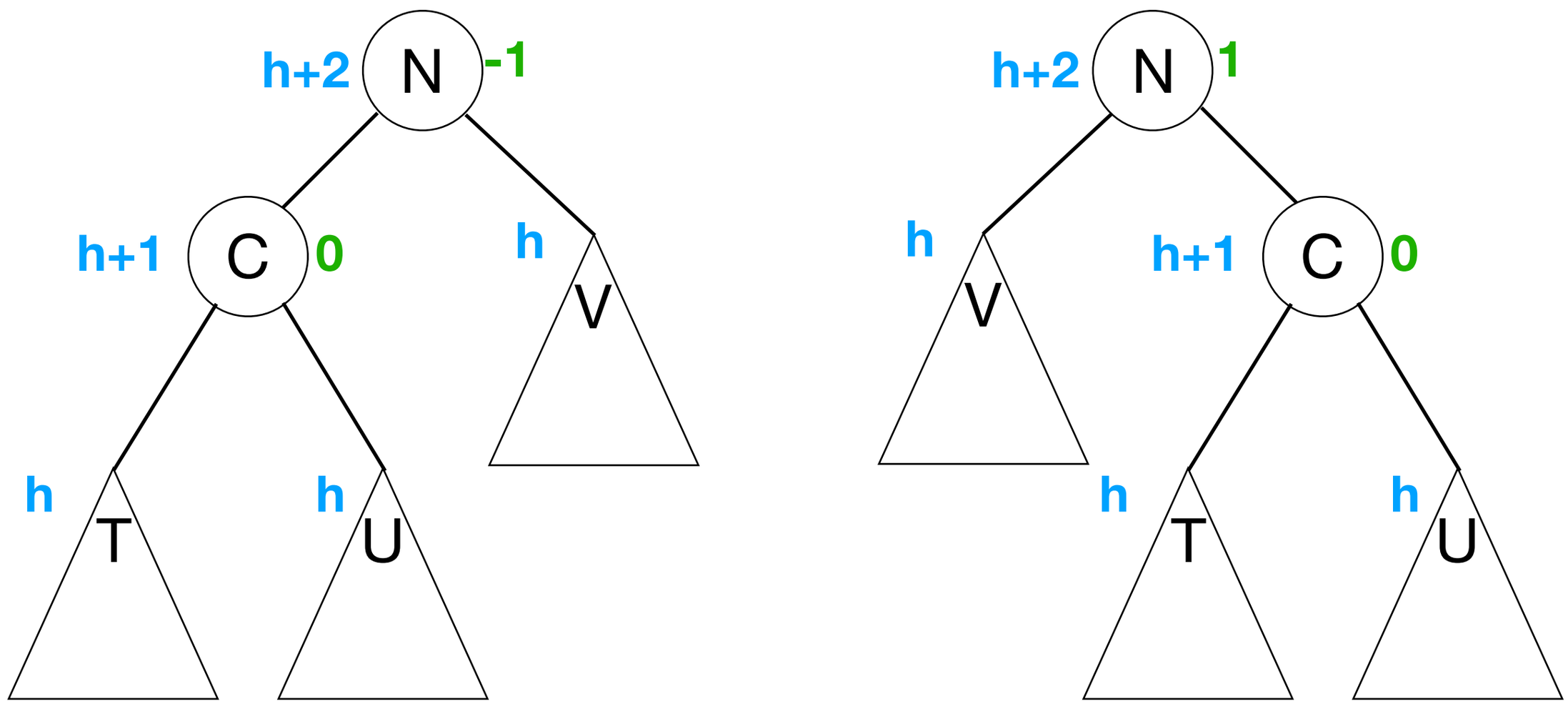
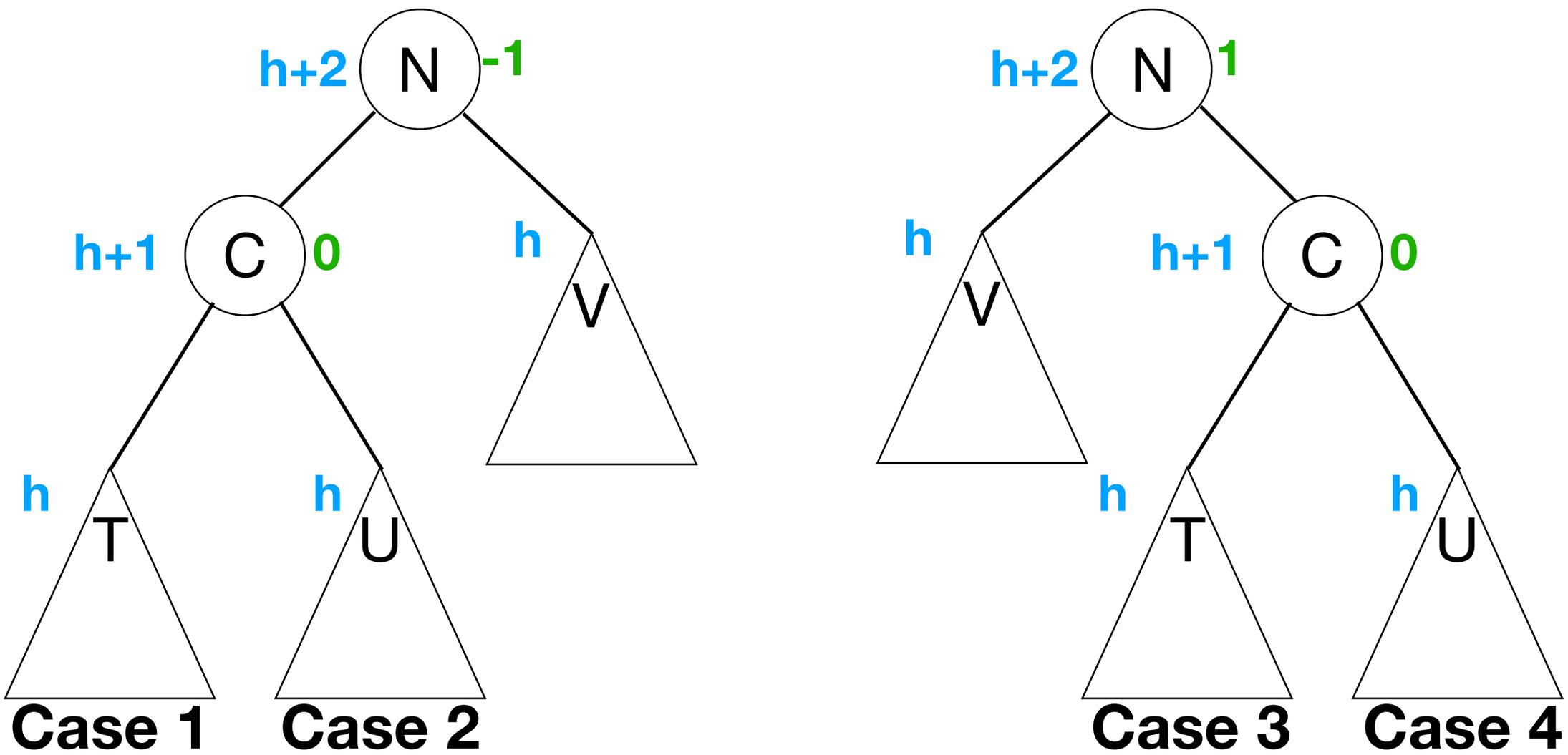


An insertion that unbalances n could go one of four places.

AVL Rebalance

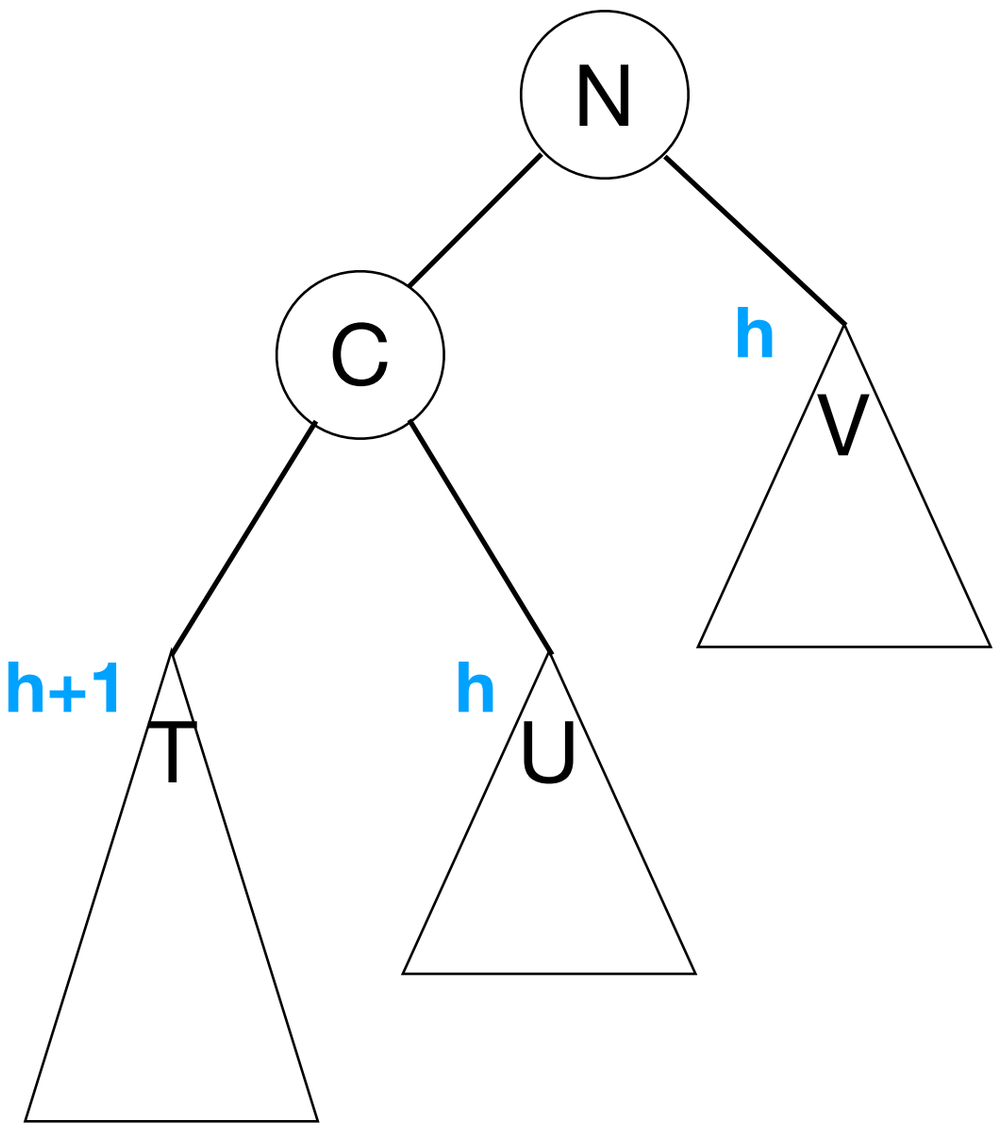
Before an insertion that unbalances n,
the tree must look like one of these:



An insertion that unbalances n could go one of four places.

AVL Rebalance

Case 1: After BST insertion step, the tree looks like this.



AVL Rebalance

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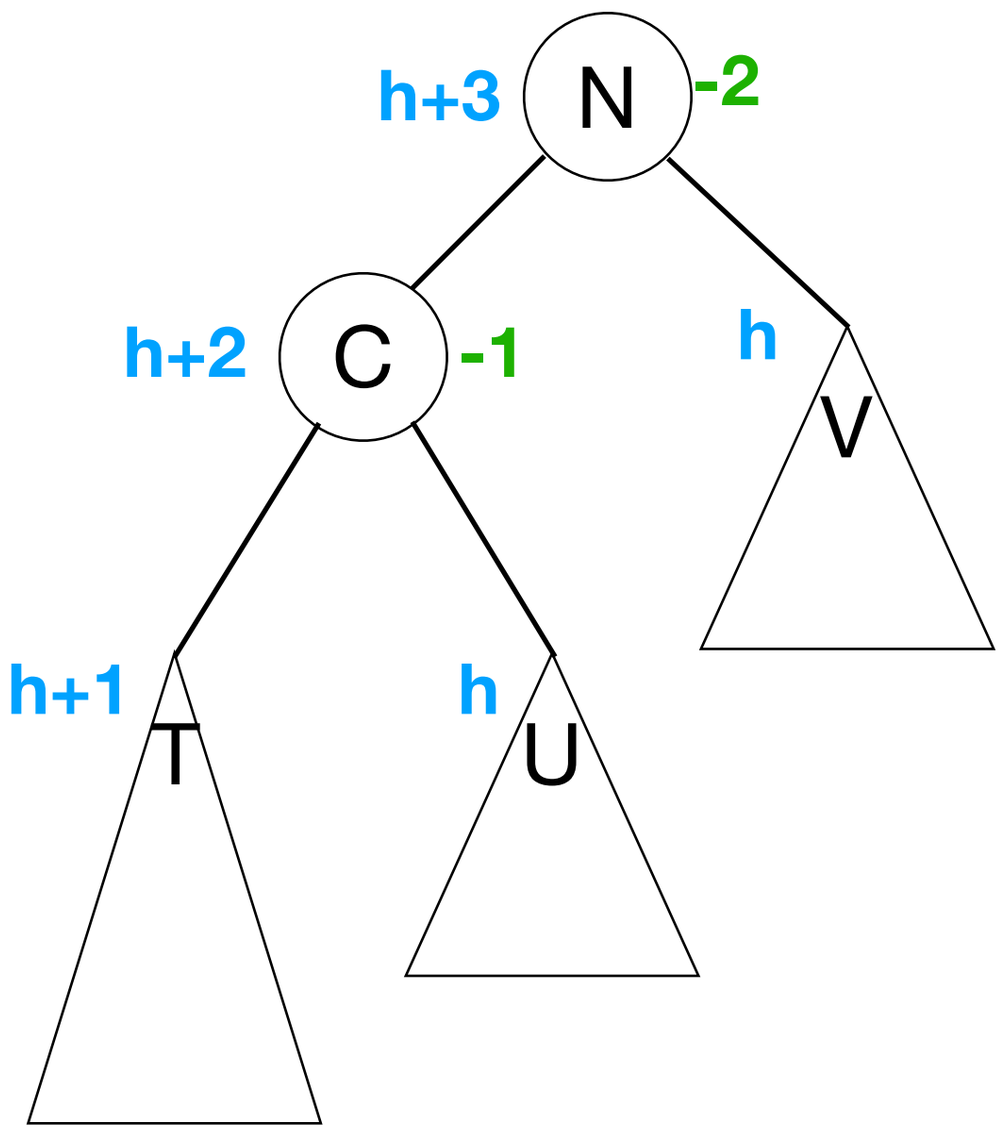
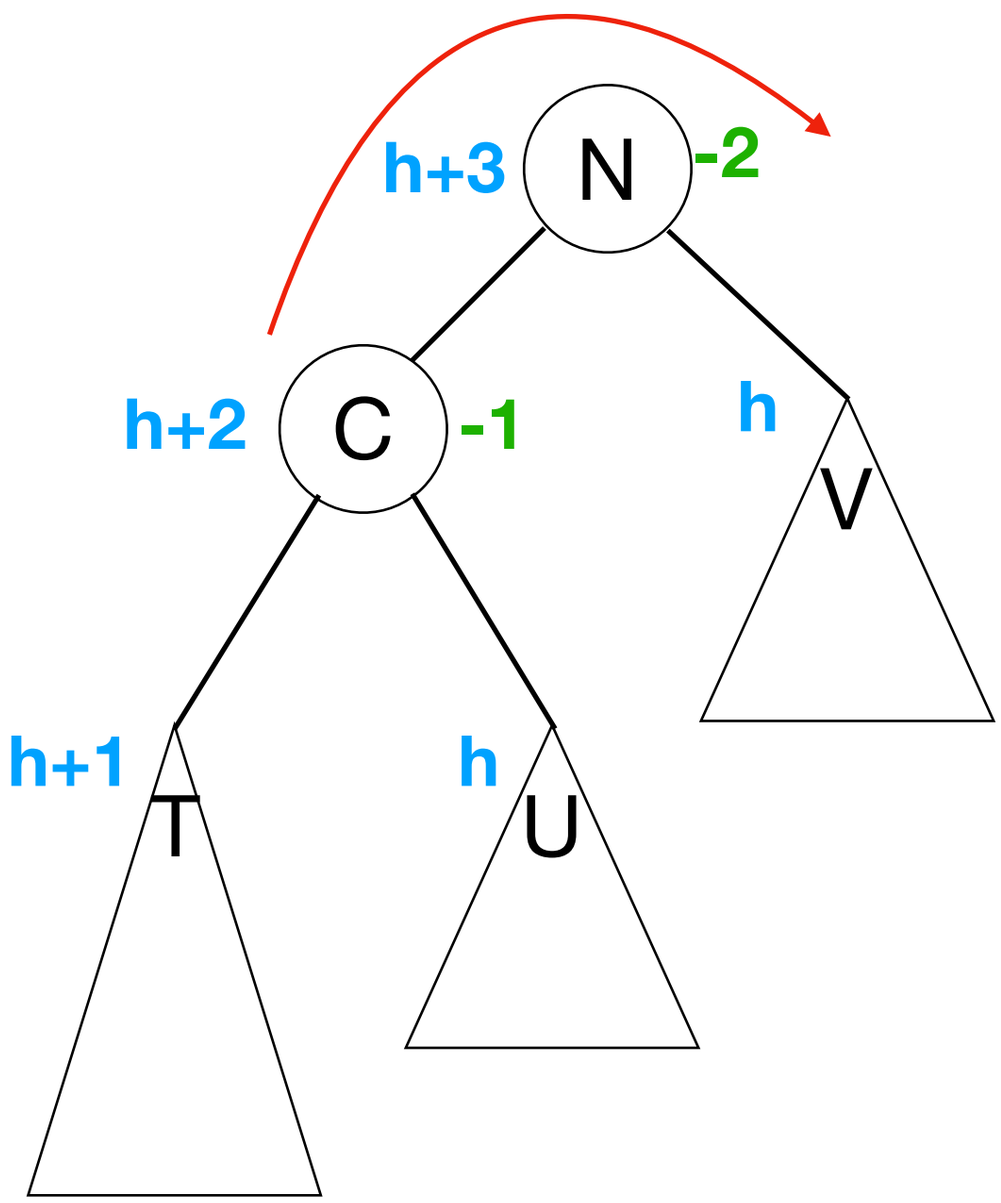


AVL Rebalance

Case 1: After BST insertion step, the tree looks like this.

Solution: right rotate on N.



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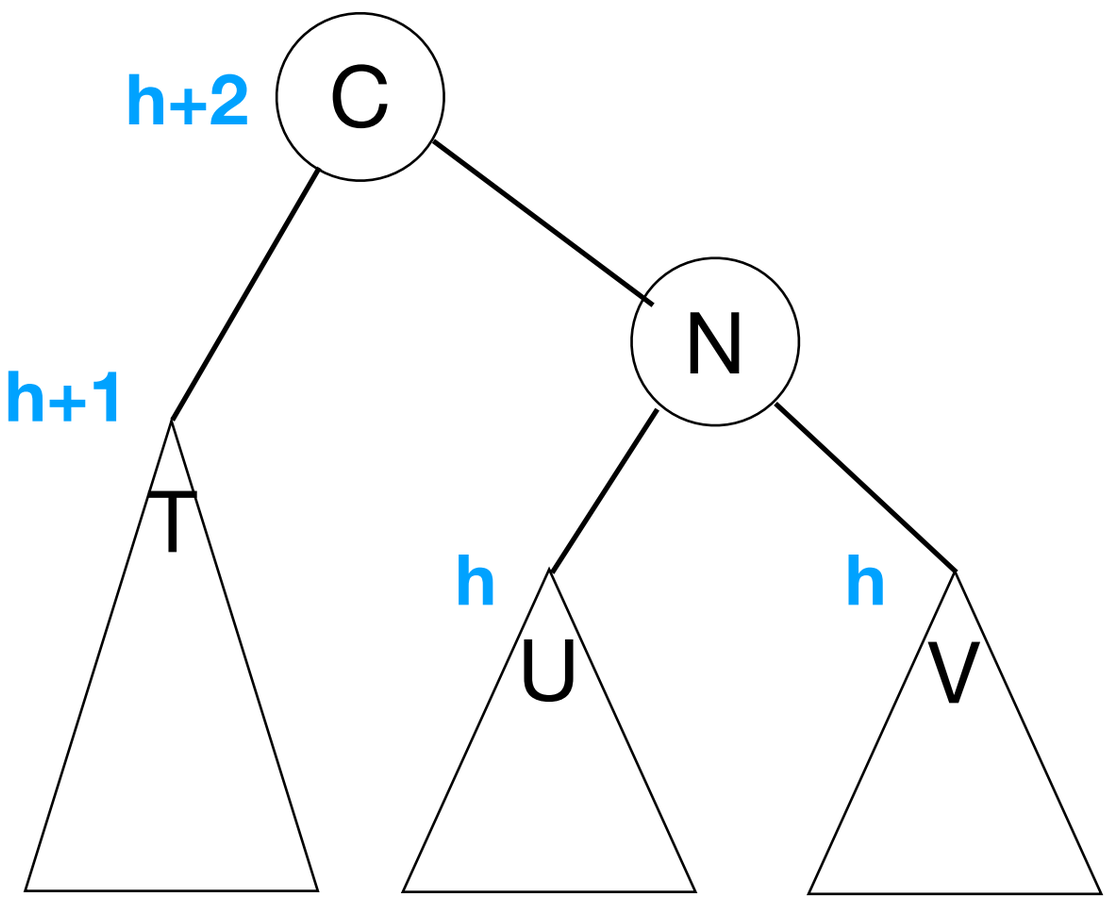
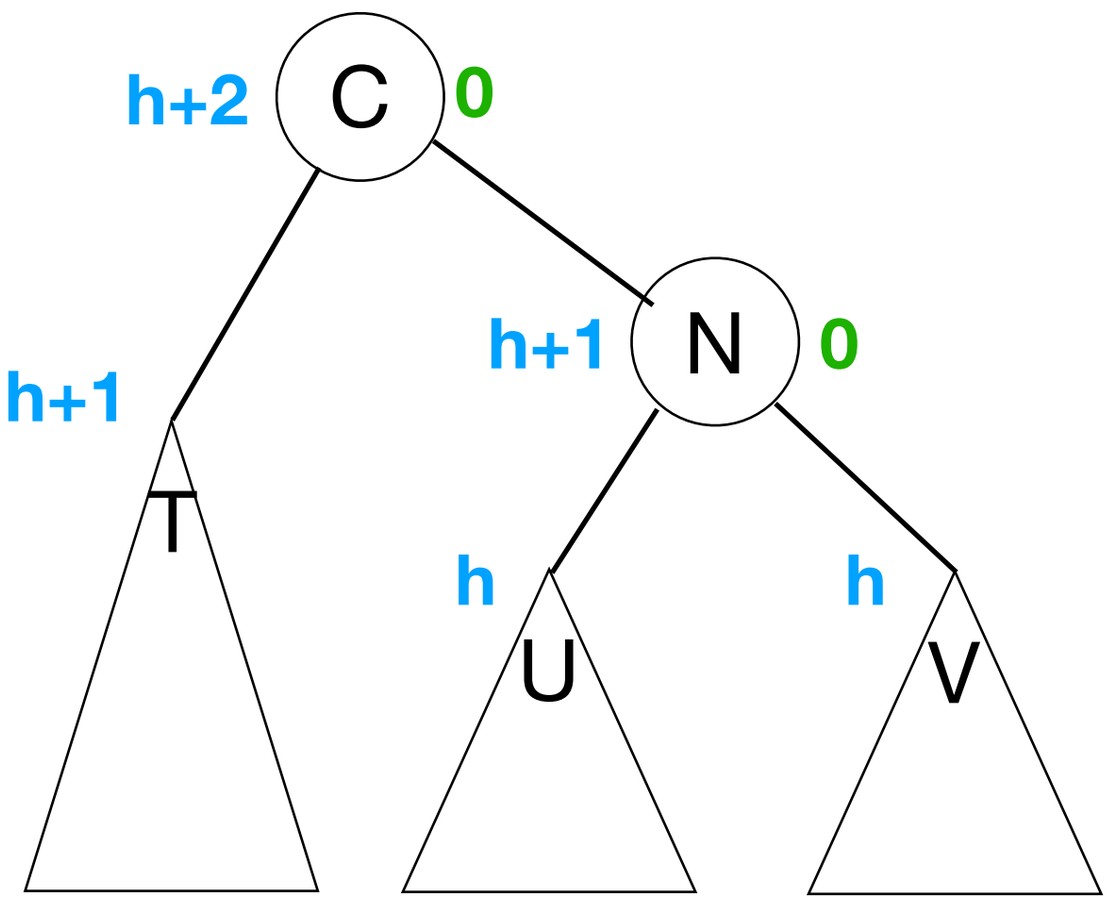


AVL Rebalance

Case 1: After BST insertion step, the tree looks like this.

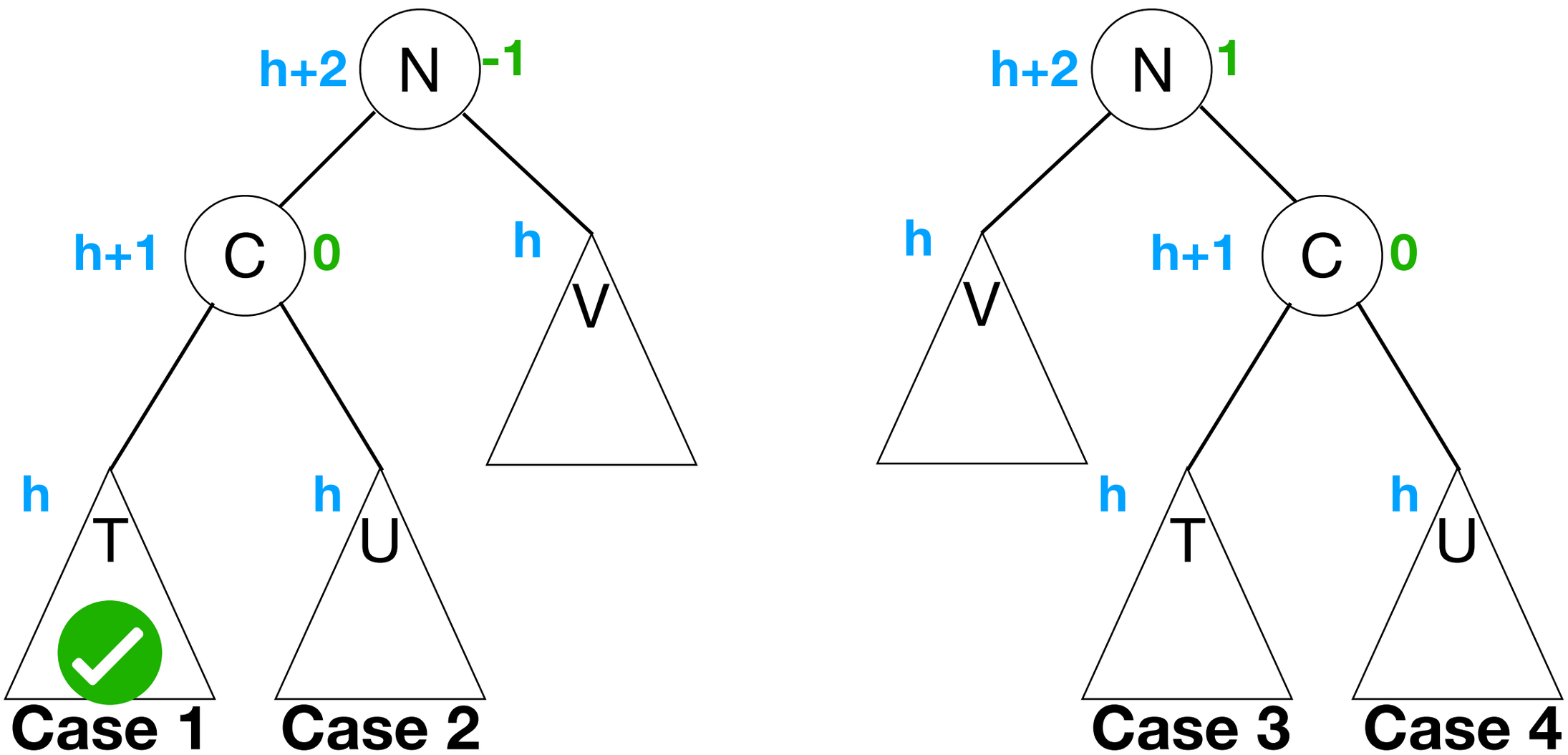
Solution: right rotate on N.

N is now AVL balanced.



AVL Rebalance

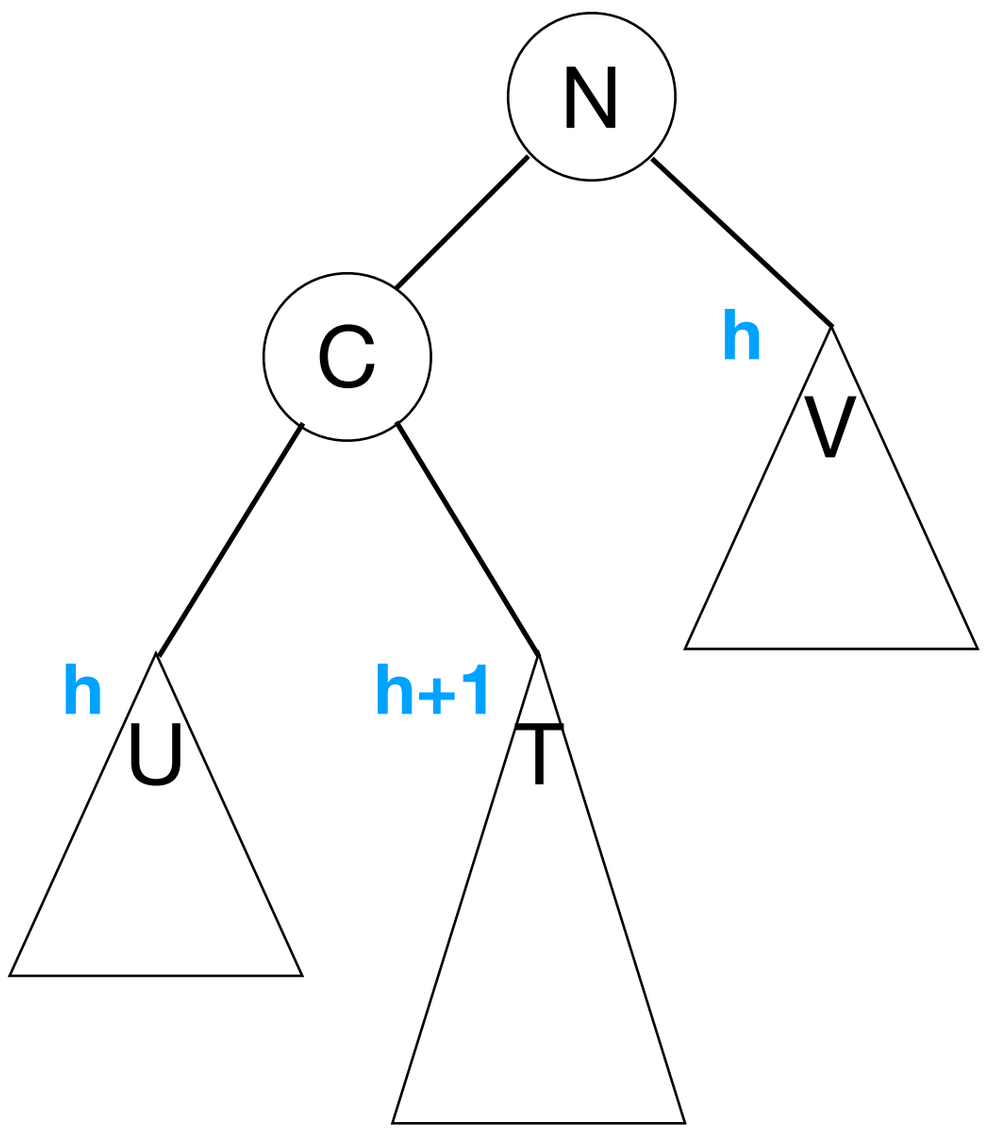
Before an insertion that unbalances n,
the tree must look like one of these:



An insertion that unbalances n could gone one of four places.

AVL Rebalance

Case 2: After BST insertion step, the tree looks like this.



AVL Rebalance

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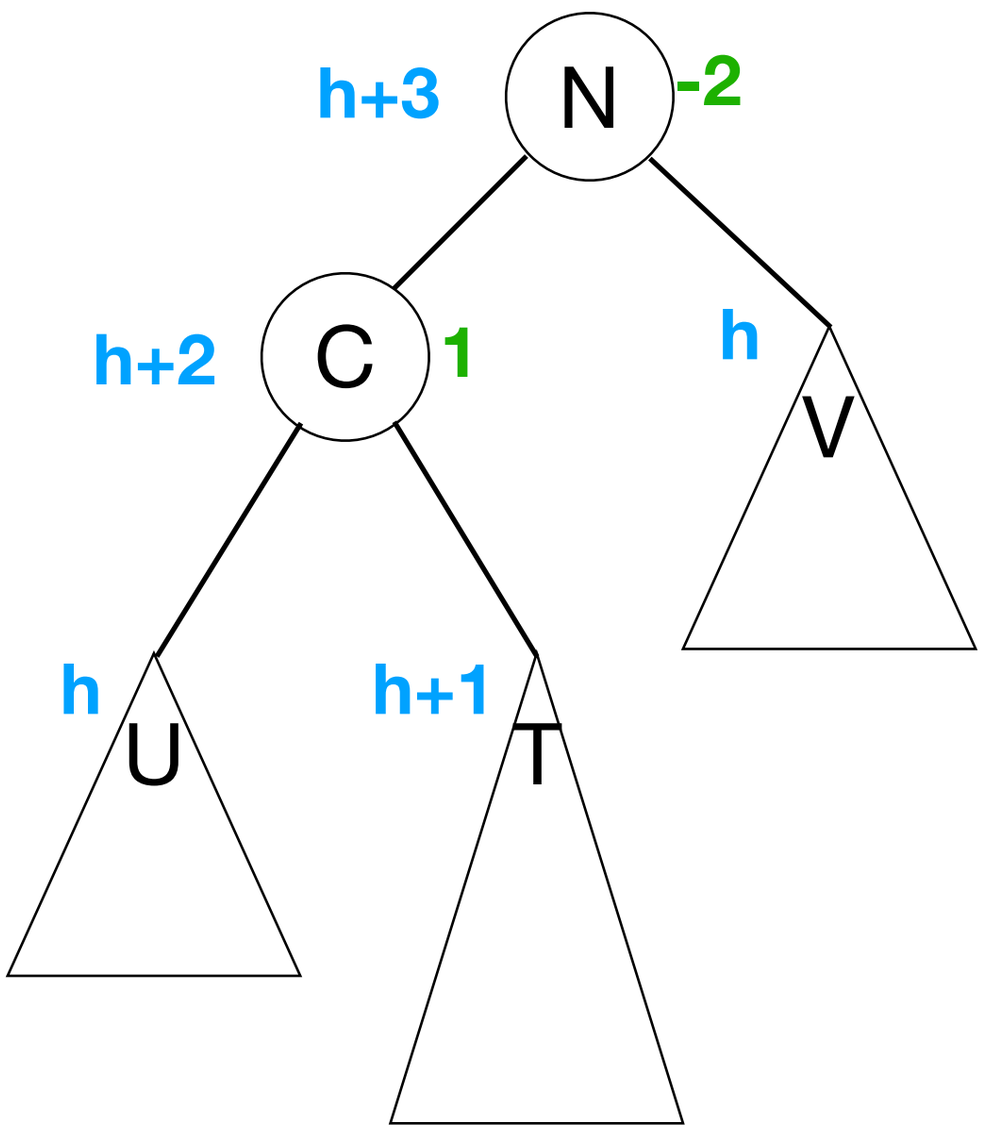

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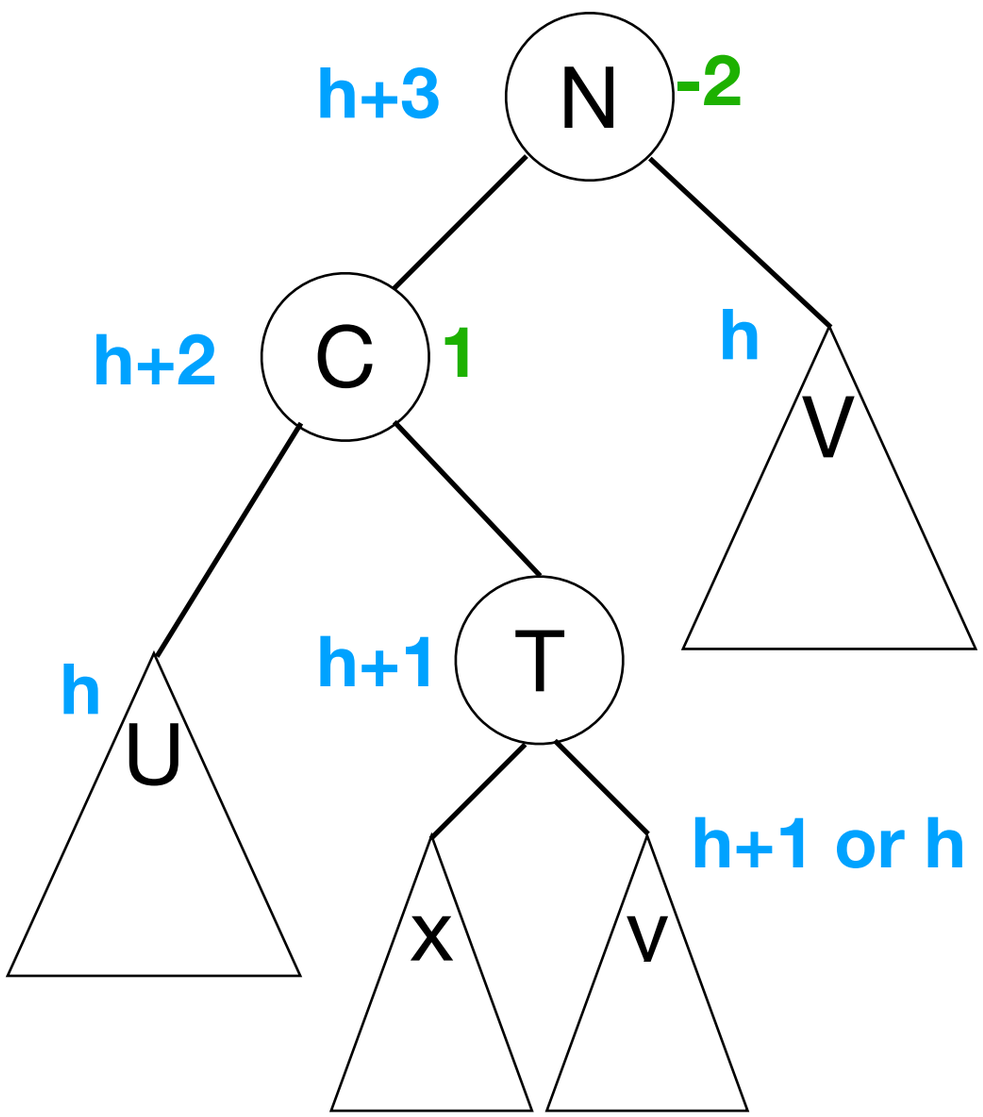
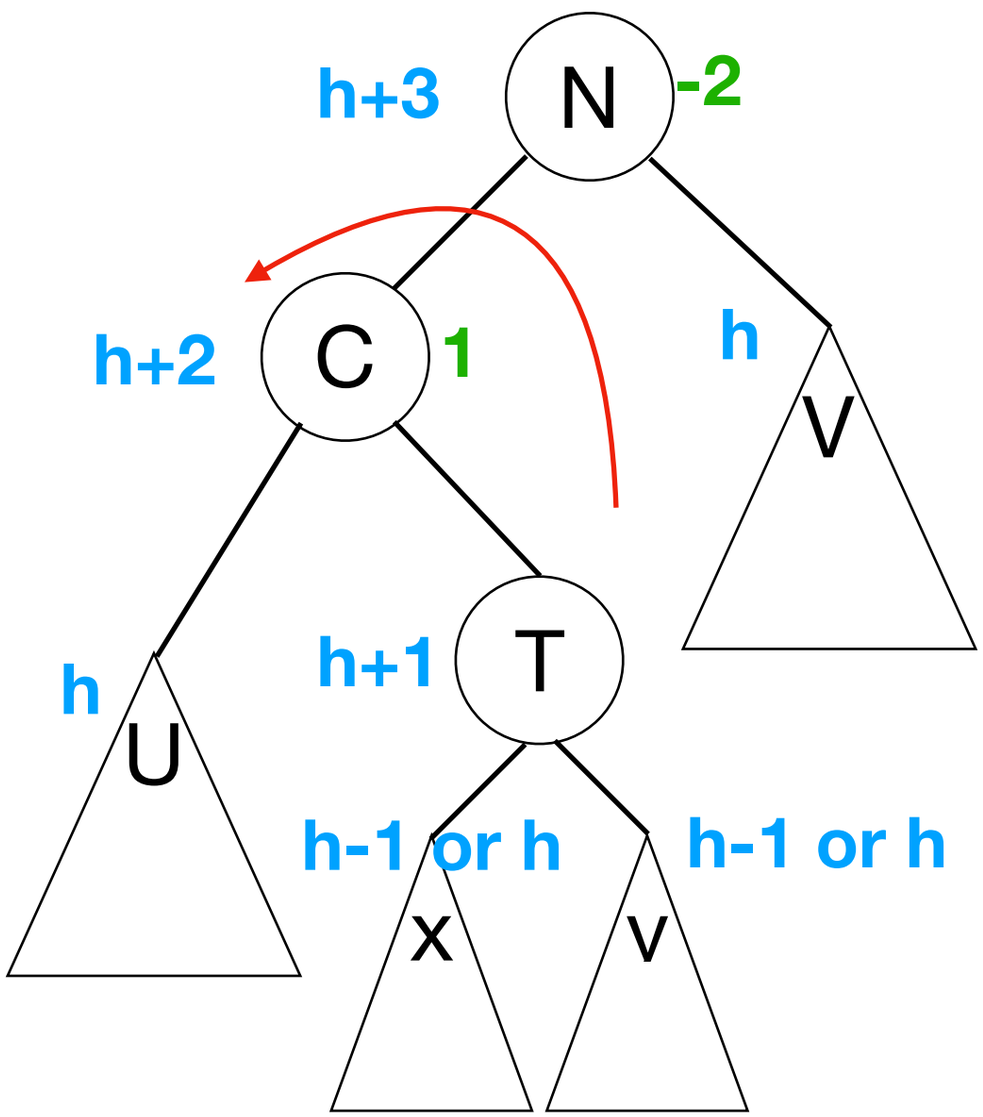


Solution - two rotations:

1. Left rotate C
2. Right rotate N

AVL Rebalance

Case 2: After BST insertion step, the tree looks like this.



Solution - two rotations:

1. **Left rotate C**
2. Right rotate N

AVL Rebalance

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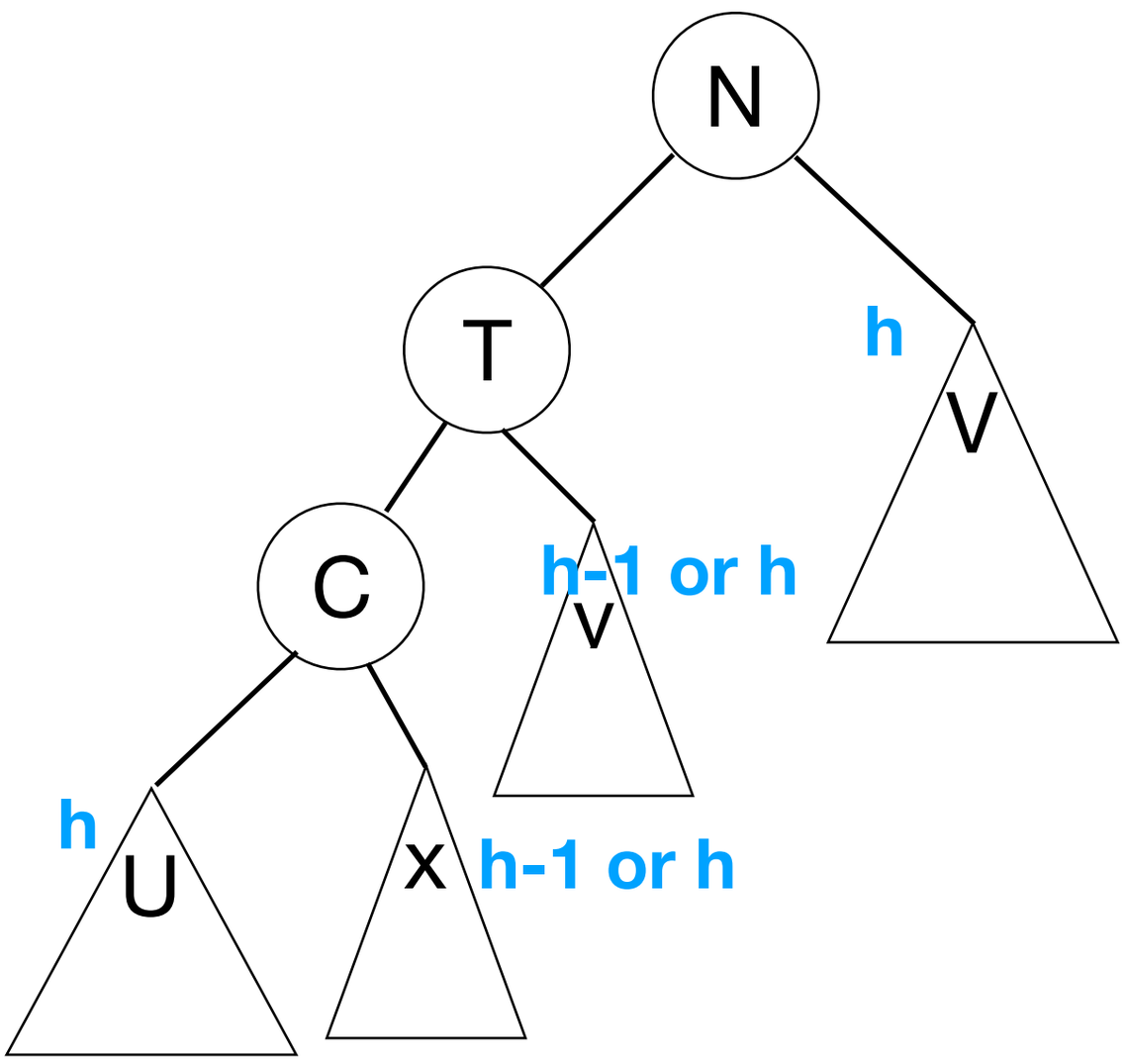

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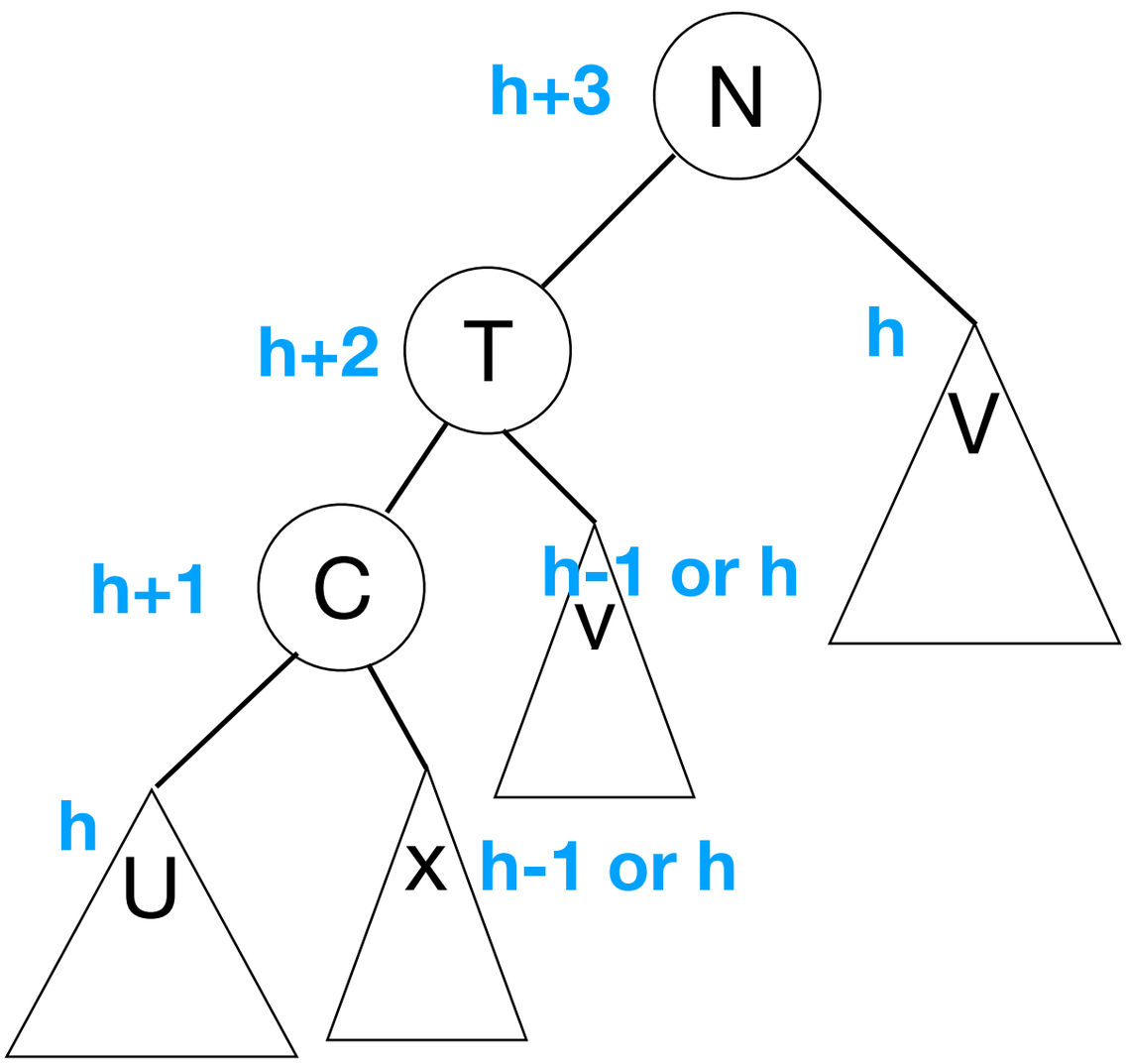

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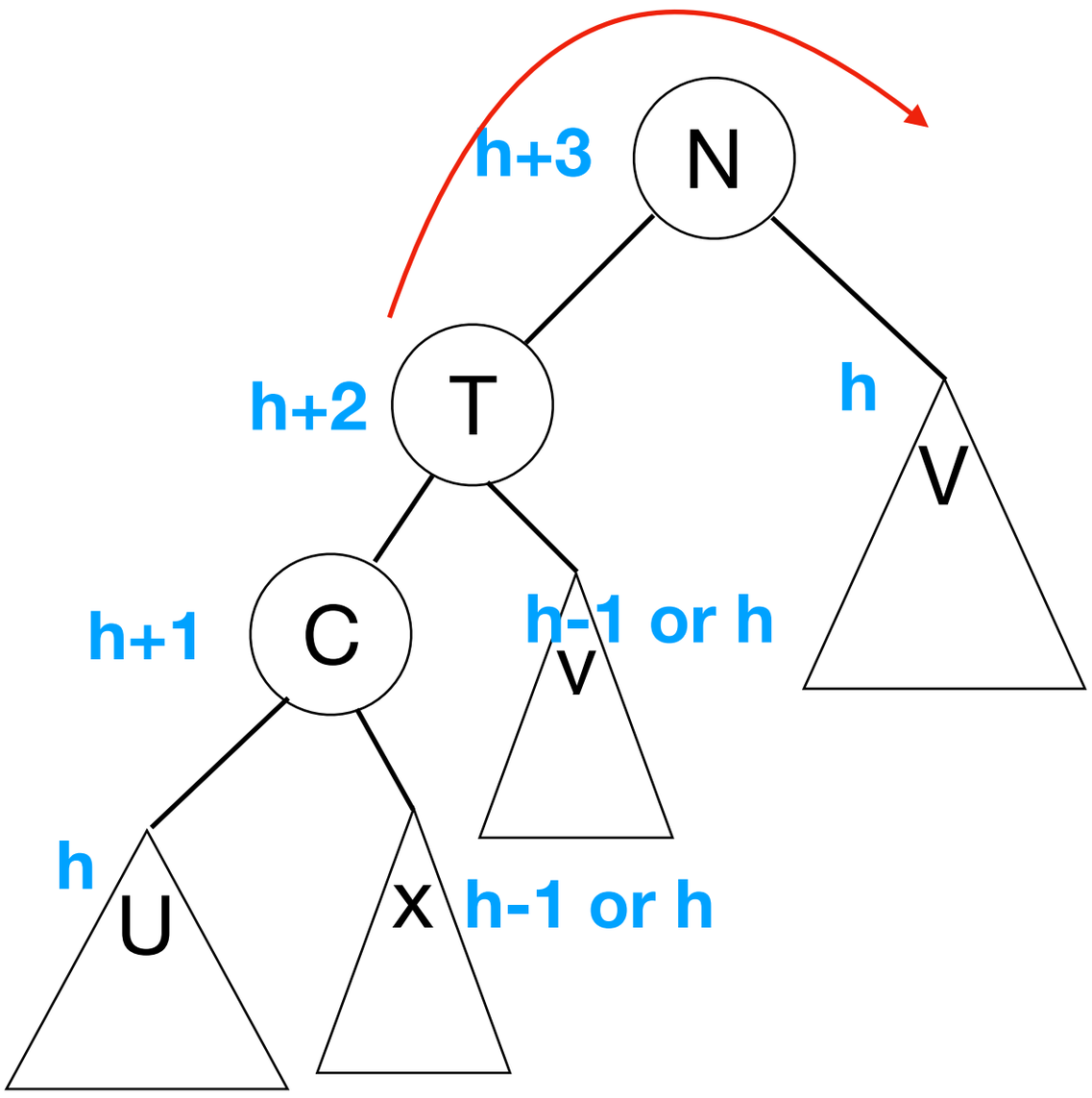

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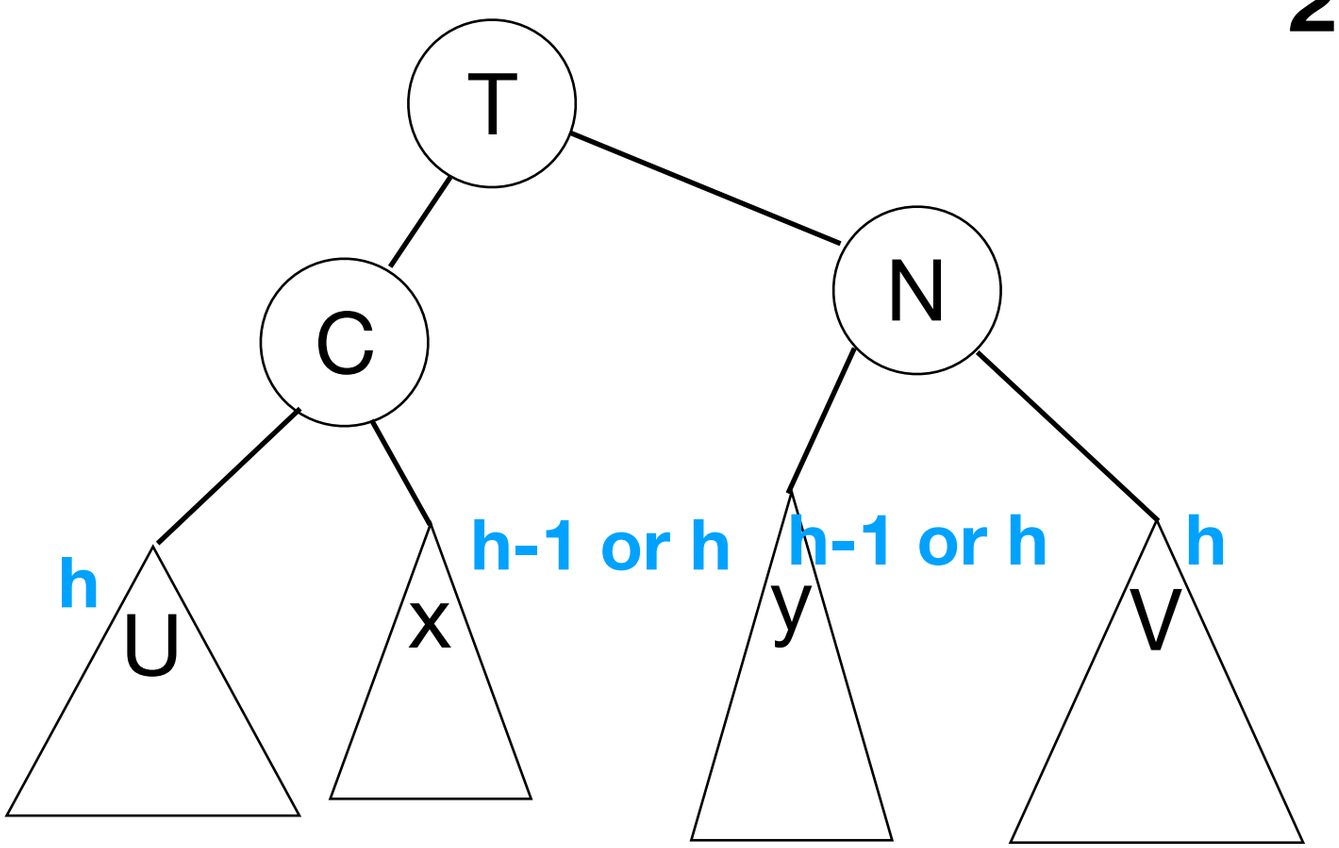

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AVL Rebalance

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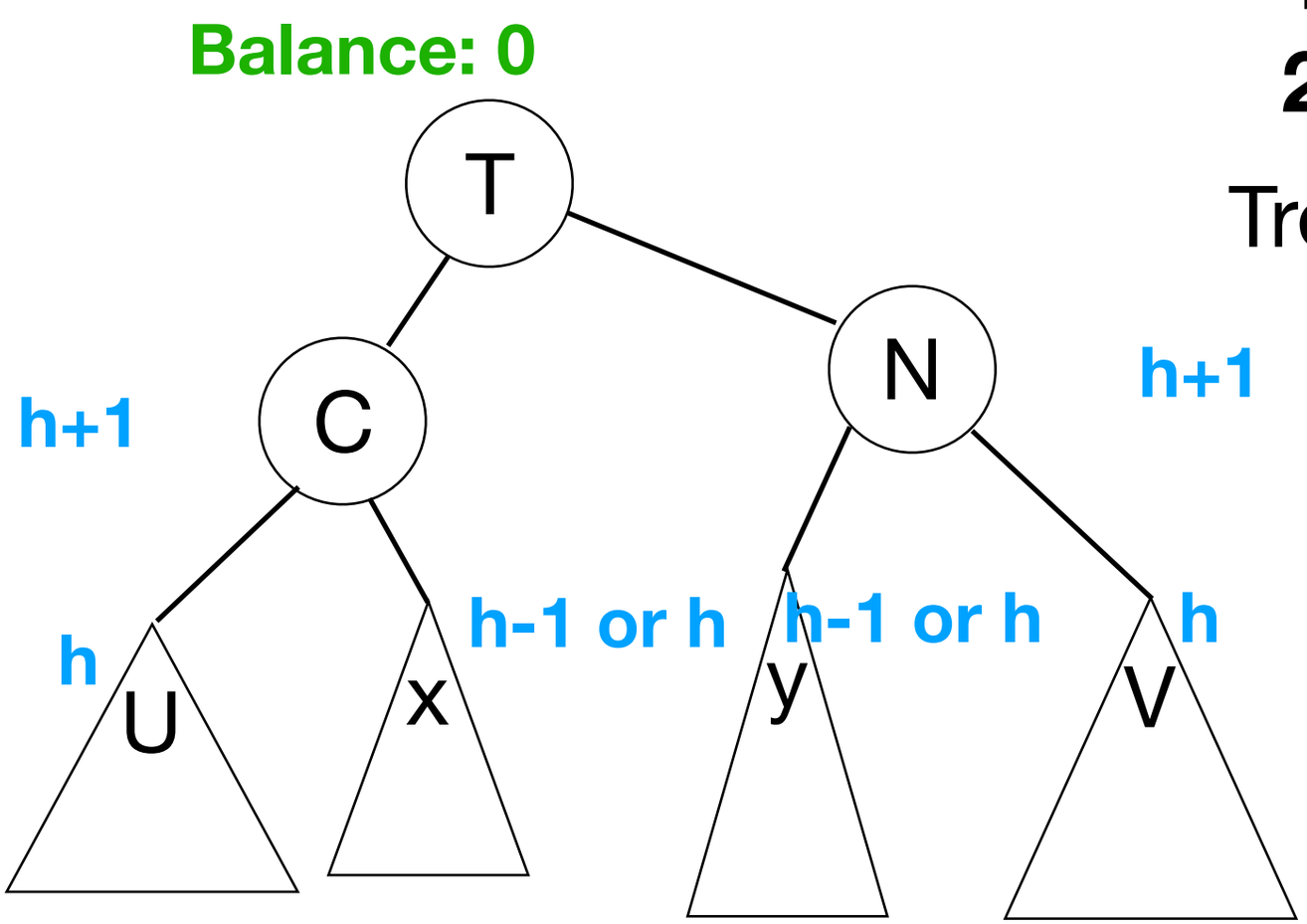


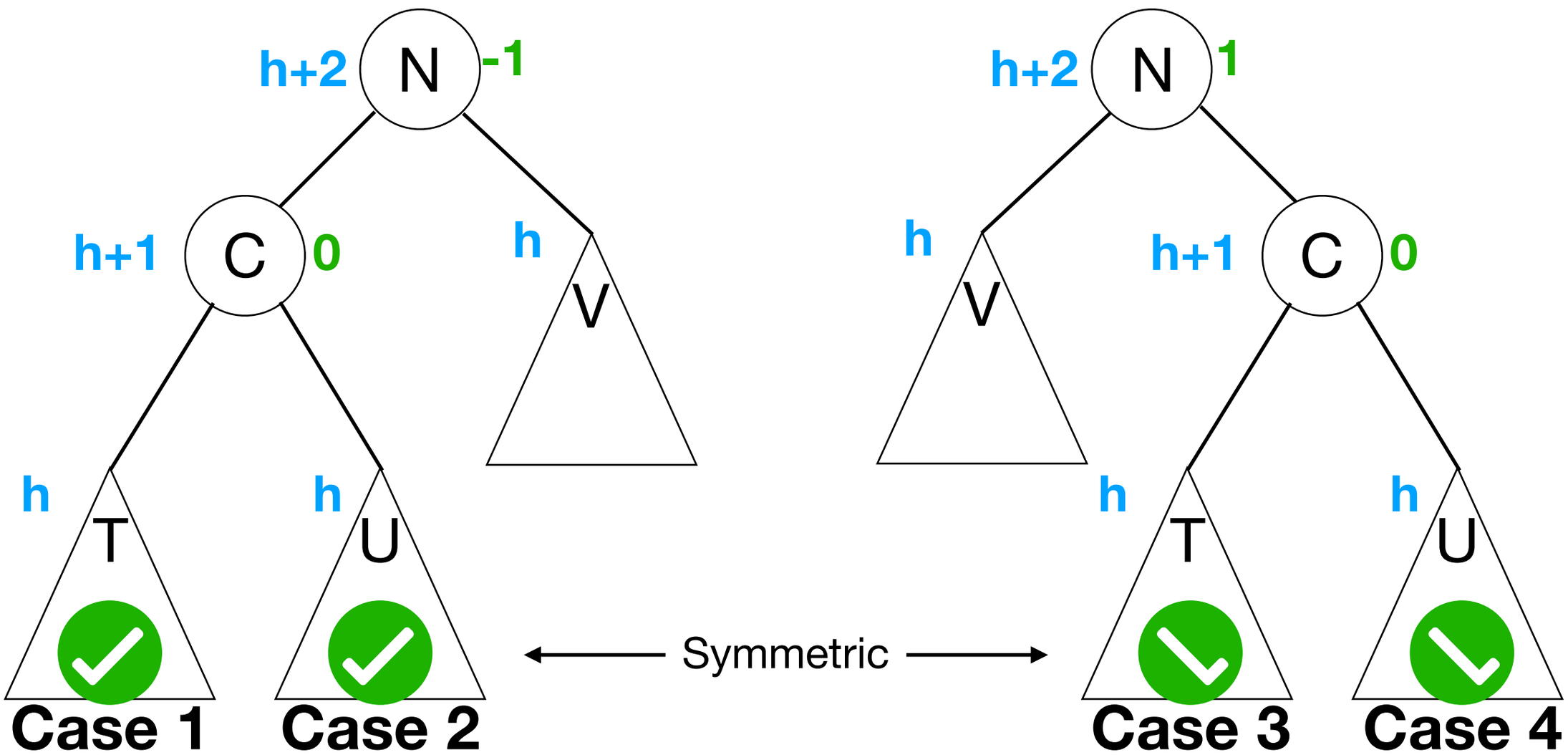
Solution - two rotations:

1. Left rotate C
2. **Right rotate N**

Tree is now AVL balanced.

AVL Rebalance

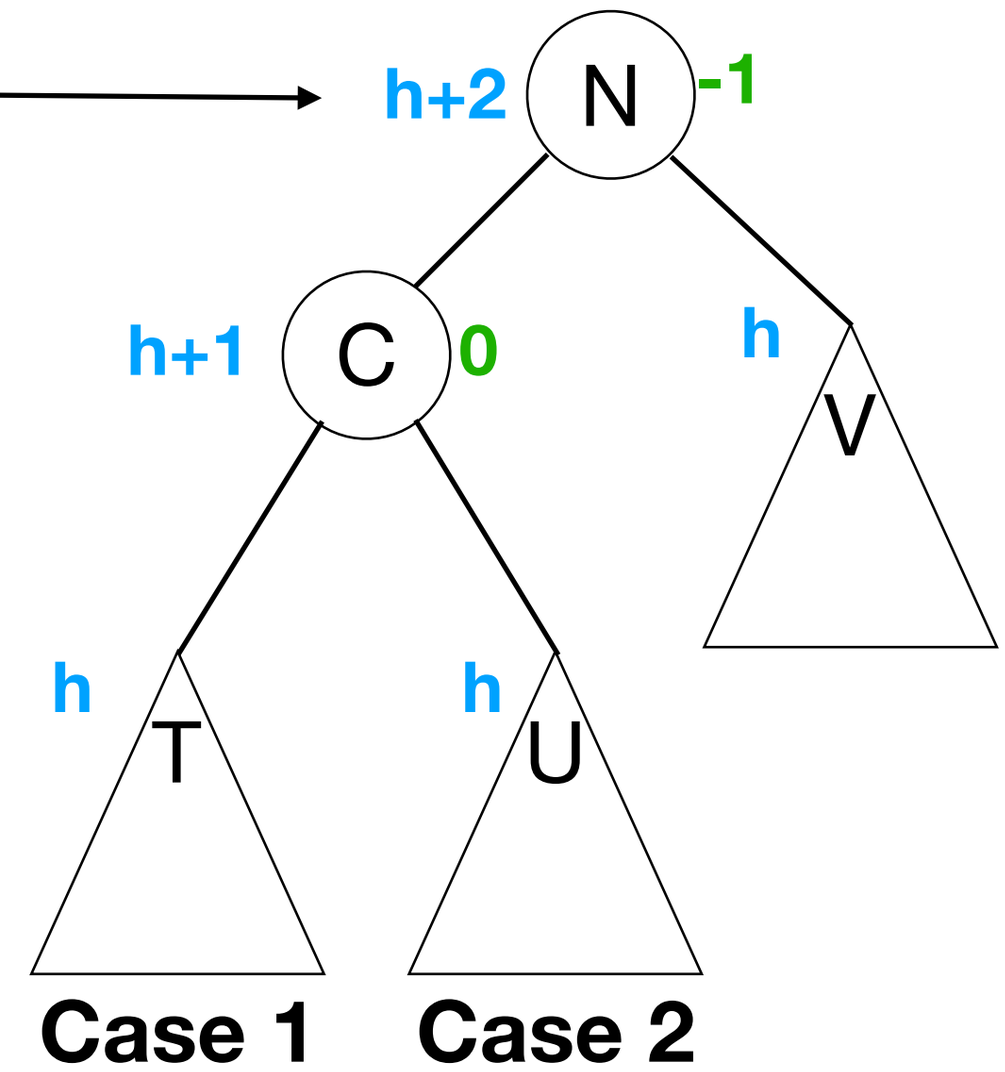
Before an insertion that unbalances n,
the tree must look like one of these:



An insertion that unbalances n could gone one of four places.

Implementation

```
void rebalance(n):
  if bal(n) < -1:
    if bal(n.left) < 0
      // case 1:
      // rightRot(n)
    else:
      // case 2:
      // leftRot(n.L);
      // rightRot(n)
  else if bal(n) > 1:
    if bal(n.right) < 0:
      // case 3:
      // rightRot(n.R);
      // leftRot(n)
    else:
      // case 4:
      // leftRot(n)
```



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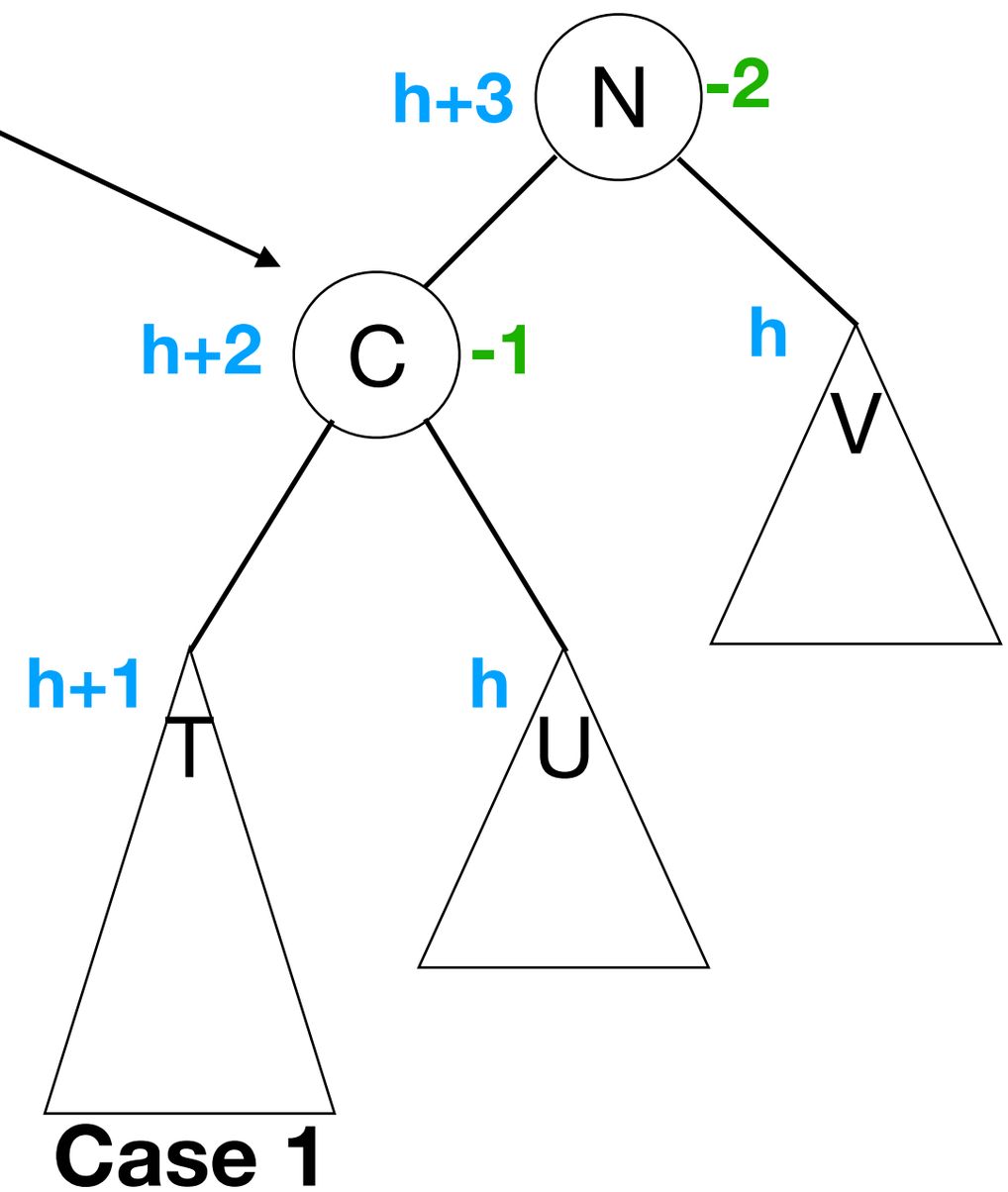
```



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      // leftRot(n)
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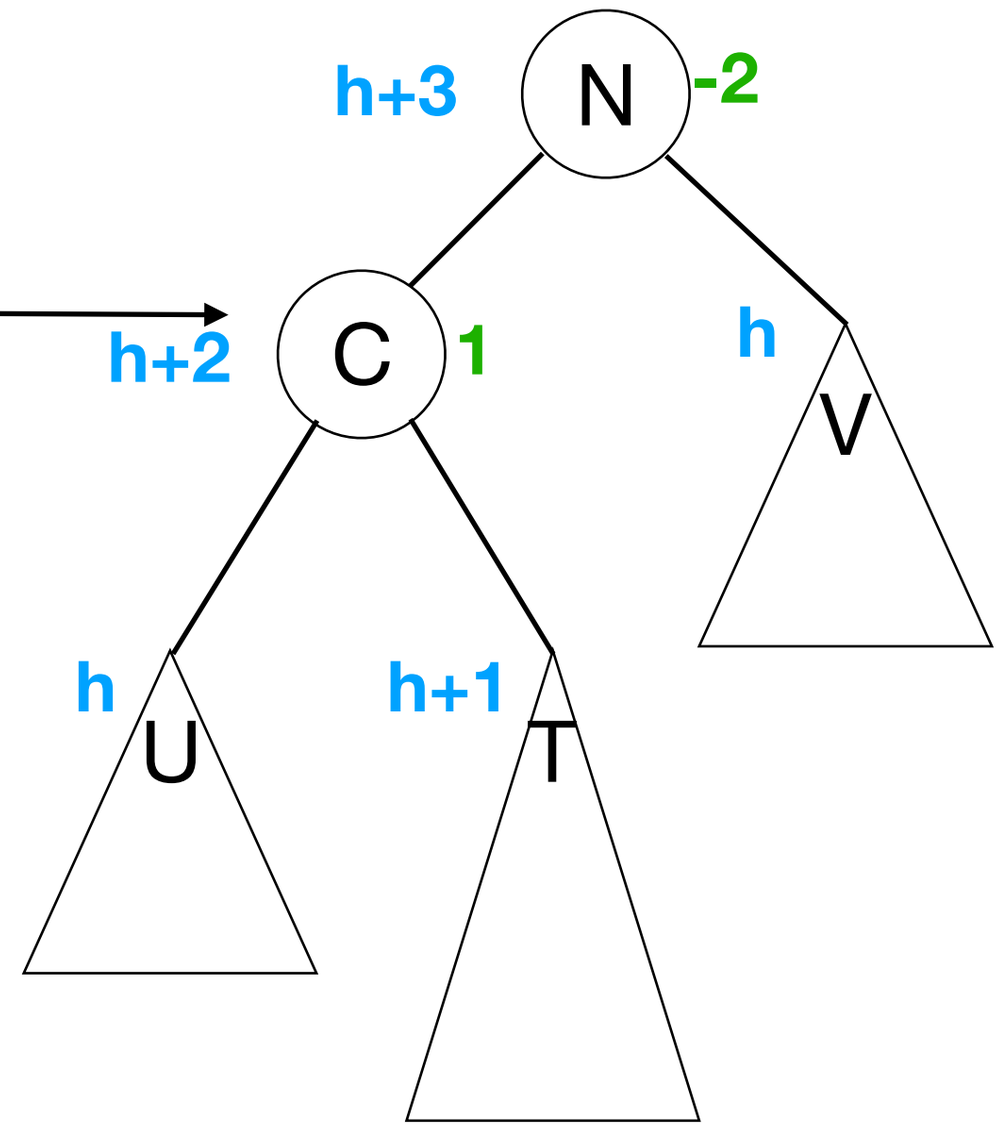
```



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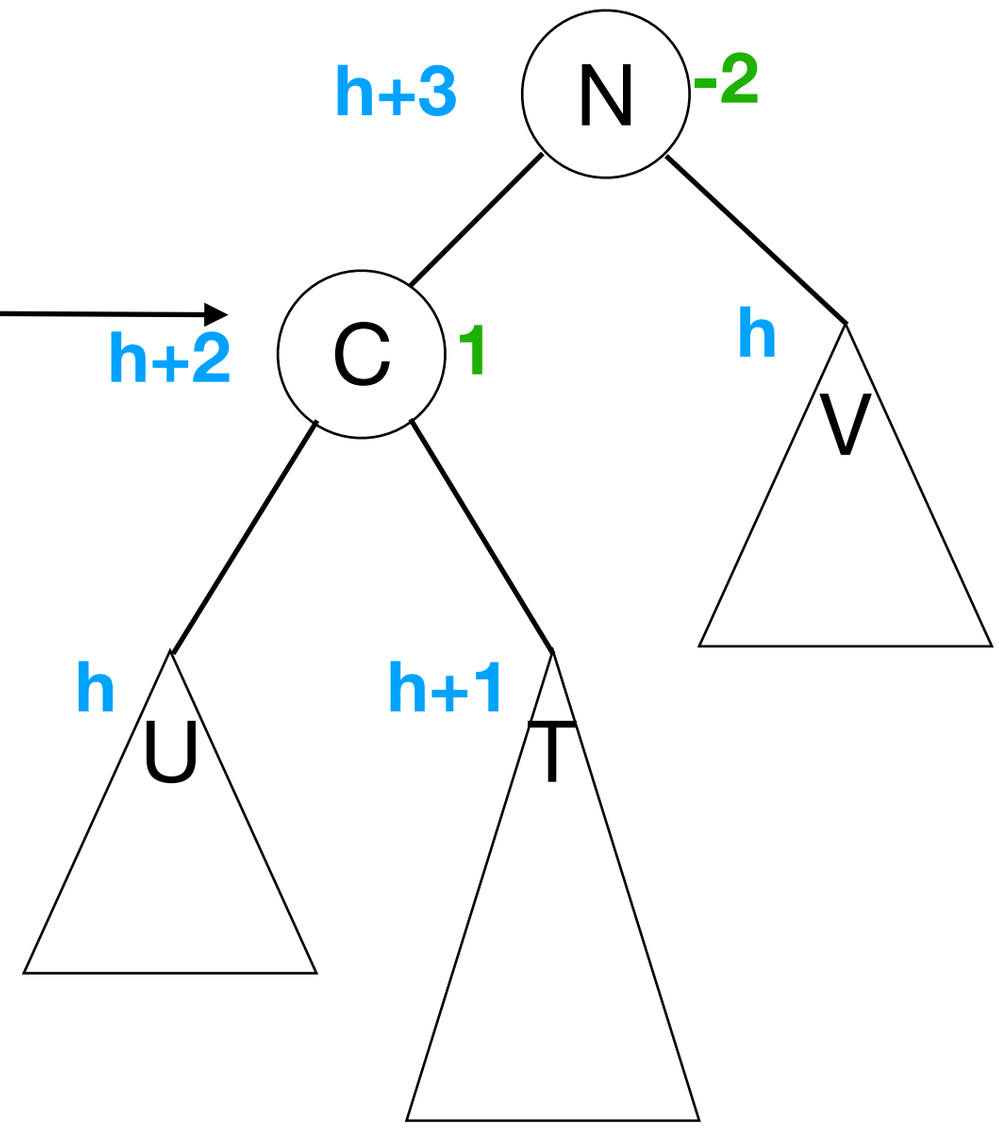
      // rightRot(n)
  else if bal(n) > 1:
    if bal(n.right) < 0:
      // case 3:
      // rightRot(n.R);
      // leftRot(n)
    else:
      // case 4:
      // leftRot(n)
```



Case 2

Cases 3 and 4 are symmetric with 2 and 1

Implementation

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    else:
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      // leftRot(n.L);
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      // case 3:
      // rightRot(n.R);
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Cases 3 and 4 are symmetric with 2 and 1.

Implementation

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      // leftRot(n.L);
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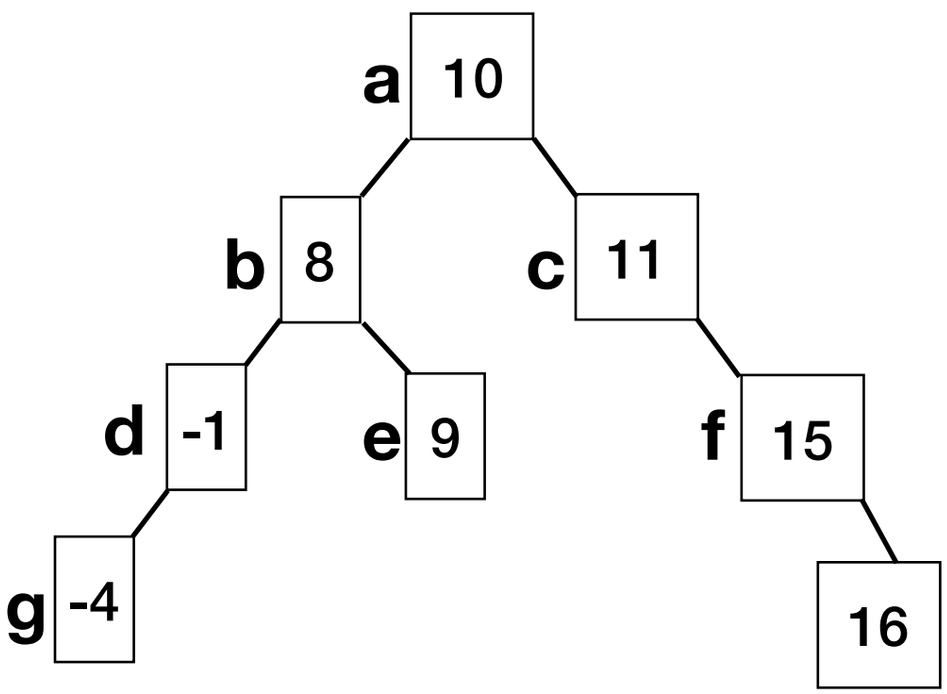
      // case 3:
      // rightRot(n.R);
      // leftRot(n)
    else:
      // case 4:
      // leftRot(n)
```

Cases 3 and 4 are symmetric with 2 and 1.

Details

- Implementing `bal`:
  - Nodes track their height
  - Need to update when the tree changes
- Edge cases near the root

Insertion with Rebalance



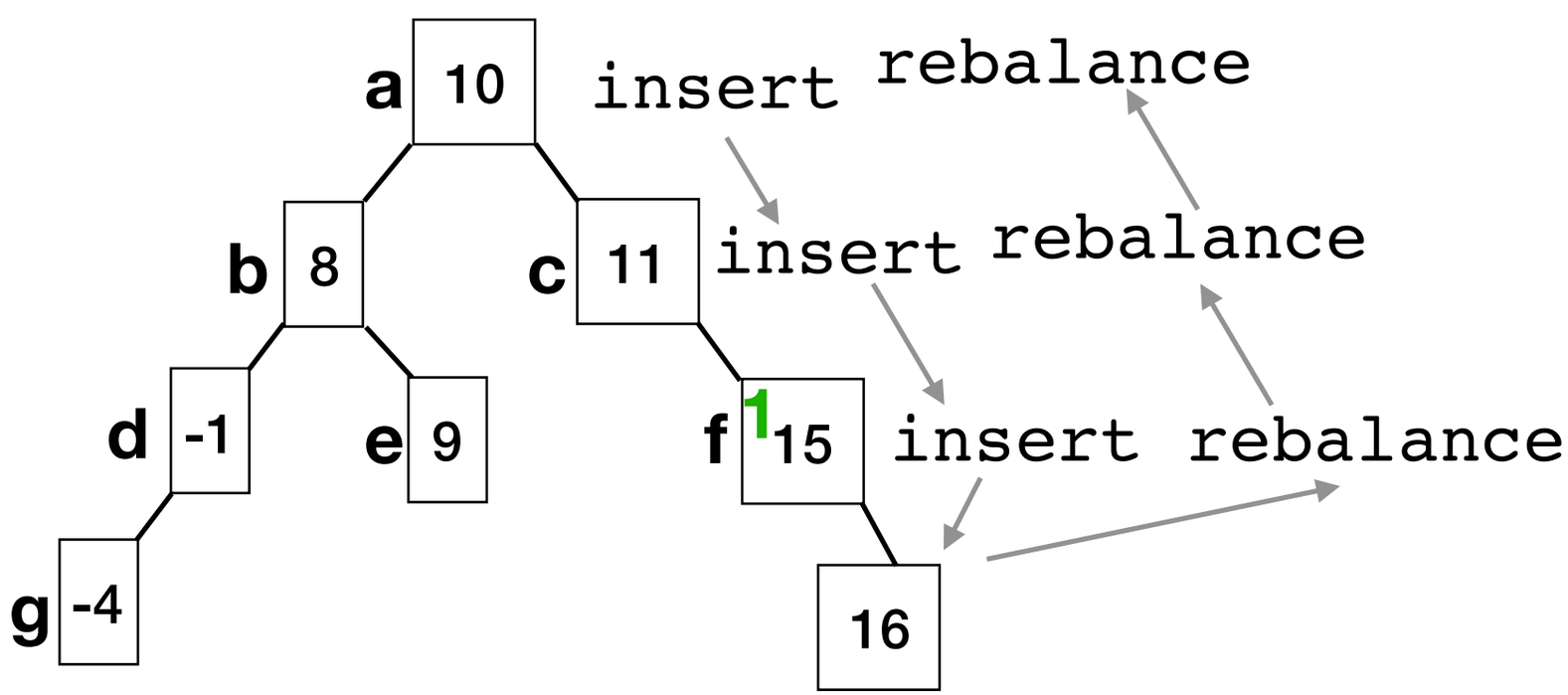

```
insert(Node n, int v):
  //…(other case, irrelevant here)
  else: // v > n.value
    if n has right:
      insert(n.right, v)
    else:
      // attach new node w/ value
      //   v to n.right
  rebalance(n);
```

How did we know what rotation to do?

```
insert(a, 16)
=>insert(c, 16)
  =>insert(f, 16)
    =>attach new node
      rebalance(f) already balanced
    rebalance(c) perform rotation
  rebalance(a) already balanced
```

Height of AVL Trees

- As usual, runtime of search, insert, and remove are all O(height).
  - A rotation is O(1), so even if we have to rebalance every node on the path to the root, it's still only h*O(1) rebalances.



Height of AVL Trees

- As usual, runtime of search, insert, and remove are all O(height)
- How many nodes in an AVL tree of height h?
- or, what's the tallest tree you can get with n nodes?
  - Exact proof involves fibonacci sequence(!)
- To add to root's height, you have to add to height of every subtree in one of one of root's subtrees.

Removing from AVL Tree

- Much like insertion: remove as usual, rebalance as necessary at each level up to the root.
- Whereas insertion only ever requires only one rebalance, deletion can require many
  - but still no more than the tree’s height.