Chapter 03: Asymptotic Growth.

Review of limits.

\[
\lim_{t \to a} f(t) = L : \forall \epsilon > 0, \exists \delta > 0, 0 < |t - a| < \delta \Rightarrow |f(t) - L| < \epsilon \\
\lim_{t \to \infty} f(t) = L : \forall \epsilon > 0, \exists T, t > T \Rightarrow |f(t) - L| < \epsilon \\
\lim_{t \to -\infty} f(t) = L : \forall \epsilon > 0, \exists T, t < T \Rightarrow |f(t) - L| < \epsilon
\]

\[
\lim_{t \to a} f(t) = \infty : \forall M, \exists \delta > 0, 0 < |t - a| < \delta \Rightarrow f(t) > M \\
\lim_{t \to \infty} f(t) = \infty : \forall M, \exists T, t > T \Rightarrow f(t) > M \\
\lim_{t \to -\infty} f(t) = \infty : \forall M, \exists T, t < T \Rightarrow f(t) > M
\]

\[
\lim_{t \to a} f(t) = -\infty : \forall M, \exists \delta > 0, 0 < |t - a| < \delta \Rightarrow f(t) < M \\
\lim_{t \to \infty} f(t) = -\infty : \forall M, \exists T, t > T \Rightarrow f(t) < M \\
\lim_{t \to -\infty} f(t) = -\infty : \forall M, \exists T, t < T \Rightarrow f(t) < M
\]

Unify with neighborhoods: \( \lim_{t \to a} f(t) = L \) means “For any neighborhood \( \mathcal{N} \) of \( L \), there is a corresponding neighborhood \( \mathcal{M} \) of \( a \) such that \( t \neq a, t \in \mathcal{M} \) implies \( f(t) \in \mathcal{N} \).” In this version either \( a \) or \( L \) or both can be \( \pm \infty \).
Examples.

1. $f(t) = \frac{200}{t}$. $\lim_{t \to \infty} f(t) = ?$. Intuitive answer: $\lim_{t \to \infty} f(t) = 0$. Formally, let $\epsilon > 0$. For $t > 0$,

$$|f(t) - 0| < \epsilon$$

$$\left| \frac{200}{t} \right| = \frac{200}{t} < \epsilon$$

$$t > \frac{200}{\epsilon}.$$

Choose $T > \frac{200}{\epsilon} > 0$. Then $t > T$ implies $t > 0$ and

$$t > T$$

$$t > \frac{200}{\epsilon}$$

$$0 < \frac{200}{t} < \epsilon$$

$$0 < f(t) < \epsilon$$

$$|f(t)| < \epsilon$$

$$|f(t) - 0| < \epsilon.$$
2. **Limits commute with arithmetic** (mostly).

\[ f(t) = \frac{4t^2 + 3t - 6}{3t^2 + 7t} \]

\[ \lim_{t \to \infty} f(t) = ? \]

\[ f(t) = \frac{4t^2 + 3t - 6}{3t^2 + 7t} = \frac{t^2(4 + 3/t - 6/t^2)}{t^2(3 + 7/t)} = \frac{4 + 3/t - 6/t^2}{3 + 7/t} \to \frac{4}{3}, \text{ as } t \to \infty \]

\[ \lim_{t \to 0} f(t) = ? \]

\[ f(t) = \frac{4t^2 + 3t - 6}{3t^2 + 7t} = \frac{4t^2 + 3t - 6}{t(3t + 7)} \]

Numerator stabilizes at \(-6\); denominator factor \((3t + 7)\) stabilizes at \(7\), but small \(t\) factor can be positive or negative. The resulting fraction takes on large positive and also large negative values in any neighborhood of zero. Consequently, no limit exists.
3. When limits do exist in contested cases \((0/0), (\pm \infty)/(\pm \infty)\), we can sometimes find it with L'Hôpital's Rule. Note: we must stop iterating L'Hôpital's Rule when the quotient gives an uncontested result.

\[
f(t) = \frac{t^2}{\ln^2 t}
\]

\[
\lim_{t \to \infty} f(t) = ?
\]

\[
\frac{t^2}{\ln^2 t} \to \frac{2t}{(2 \ln t)(1/t)} = \frac{t^2}{\ln t} \to \frac{2t}{(1/t)} = 2t^2 \to \infty.
\]
Asymptotic classes

We will deal exclusively with functions of the form $f : \{1, 2, \ldots\} \to (0, \infty)$. These functions arise from instruction counts in algorithms.

Each function $g$ defines five asymptotic classes: $o(g), O(g), \Theta(g), \Omega(g), \omega(g)$. These are classes of functions related to $g$ in specific ways.

Definitions.

- $o(g)$: $f \in o(g)$, also written $f = o(g)$, if $\lim_{n \to \infty} f(n)/g(n) = 0$.
- $O(g)$: $f \in O(g)$, also written $f = O(g)$, if there exist integer $N$ and $K < \infty$ such that $n \geq N$ implies $f(n)/g(n) \leq K$.
- $\Theta(g)$: $f \in \Theta(g)$, also written $f = \Theta(g)$, if there exist integer $N$, $K_1 > 0$, and $K_2 < \infty$ such that $n \geq N$ implies $K_1 \leq f(n)/g(n) \leq K_2$.
- $\Omega(g)$: $f \in \Omega(g)$, also written $f = \Omega(g)$, if there exist integer $N$ and $K > 0$ such that $n \geq N$ implies $K \leq f(n)/g(n)$.
- $\omega(g)$: $f \in \omega(g)$, also written $f = \omega(g)$, if $\lim_{n \to \infty} f(n)/g(n) = \infty$.

We refer to $f(n)/g(n)$ as “the ratio of the candidate to the reference” or simply “the ratio.”
Definitions.

\( o(g): f \in o(g), \) also written \( f = o(g), \) if \( \lim_{n \to \infty} \frac{f(n)}{g(n)} = 0. \)

\( O(g): f \in O(g), \) also written \( f = O(g), \) if there exist integer \( N \) and \( K < \infty \) such that \( n \geq N \) implies \( \frac{f(n)}{g(n)} \leq K. \)

\( \Theta(g): f \in \Theta(g), \) also written \( f = \Theta(g), \) if there exist integer \( N, K_1 > 0, \) and \( K_2 < \infty \) such that \( n \geq N \) implies \( K_1 \leq \frac{f(n)}{g(n)} \leq K_2. \)

\( \Omega(g): f \in \Omega(g), \) also written \( f = \Omega(g), \) if there exist integer \( N \) and \( K > 0 \) such that \( n \geq N \) implies \( K \leq \frac{f(n)}{g(n)}. \)

\( \omega(g): f \in \omega(g), \) also written \( f = \omega(g), \) if \( \lim_{n \to \infty} \frac{f(n)}{g(n)} = \infty. \)

Observation.

\( o(g) \subseteq O(g). \)  Why?  \( o(g) \) is where the ratio approaches zero, which means it is particularly easy to get a constant upper bound that is valid for all large \( n. \)
Definitions.

\( o(g) \): \( f \in o(g) \), also written \( f = o(g) \), if \( \lim_{n \to \infty} f(n)/g(n) = 0 \).

\( O(g) \): \( f \in O(g) \), also written \( f = O(g) \), if there exist integer \( N \) and \( K < \infty \) such that \( n \geq N \) implies \( f(n)/g(n) \leq K \).

\( \Theta(g) \): \( f \in \Theta(g) \), also written \( f = \Theta(g) \), if there exist integer \( N, K_1 > 0 \), and \( K_2 < \infty \) such that \( n \geq N \) implies \( K_1 \leq f(n)/g(n) \leq K_2 \).

\( \Omega(g) \): \( f \in \Omega(g) \), also written \( f = \Omega(g) \), if there exist integer \( N \) and \( K > 0 \) such that \( n \geq N \) implies \( K \leq f(n)/g(n) \).

\( \omega(g) \): \( f \in \omega(g) \), also written \( f = \omega(g) \), if \( \lim_{n \to \infty} f(n)/g(n) = \infty \).

Observation.

\( \Theta(g) \subseteq O(g) \) and \( \Theta(g) \subseteq \Omega(g) \). Why? \( O(g) \) requires an upper bound on the ratio, \( \Omega(g) \) requires a lower bound, \( \Theta(g) \) requires both.
Definitions.

\( o(g) \): \( f \in o(g) \), also written \( f = o(g) \), if \( \lim_{n \to \infty} \frac{f(n)}{g(n)} = 0 \).

\( O(g) \): \( f \in O(g) \), also written \( f = O(g) \), if there exist integer \( N \) and \( K < \infty \) such that \( n \geq N \) implies \( \frac{f(n)}{g(n)} \leq K \).

\( \Theta(g) \): \( f \in \Theta(g) \), also written \( f = \Theta(g) \), if there exist integer \( N, K_1 > 0 \), and \( K_2 < \infty \) such that \( n \geq N \) implies \( K_1 \leq \frac{f(n)}{g(n)} \leq K_2 \).

\( \Omega(g) \): \( f \in \Omega(g) \), also written \( f = \Omega(g) \), if there exist integer \( N \) and \( K > 0 \) such that \( n \geq N \) implies \( K \leq \frac{f(n)}{g(n)} \).

\( \omega(g) \): \( f \in \omega(g) \), also written \( f = \omega(g) \), if \( \lim_{n \to \infty} \frac{f(n)}{g(n)} = \infty \).

Observation.

\( \omega(g) \subseteq \Omega(g) \). Why? \( \omega(g) \) is where the ratio approaches \( \infty \), which means it is particularly easy to get a constant, non-zero lower bound.
We have shown that

\[
\begin{align*}
o(g) & \subseteq O(g) \\
\Theta(g) & \subseteq O(g) \\
\Theta(g) & \subseteq \Omega(g) \\
\omega(g) & \subseteq \Omega(g).
\end{align*}
\]

**Observation.**

Each reference \( g \) segregates functions into six compartments as shown below for \( g(n) = n^2 \).

In \( \Theta(n^2) \), we find \( n^2 \), as expected, but also \( n^2 + n \log n \).

In \( O(n^2) \), we find \( n, n \log n, \sqrt{n}, n^{3/2} \), and so forth.

In \( \Omega(n^2) \), we find \( n^3, n^2 \log n, n^{5/2} \), and so forth.
For areas A, B, C, we need to be more inventive.

For area A, we need a finite upper bound on the ratio,

but there cannot be a non-zero lower bound.

Moreover the ratio cannot approach zero.

One method to satisfy all these constraints is to define a function with varying behavior as its argument goes to infinity. For example $f(n) \in A$, where

$$f(n) = \begin{cases} n^2, & n \text{ even} \\ n, & n \text{ odd}. \end{cases}$$
Similarly, area B admits

\[ f(n) = \begin{cases} 
  n^2, & n \text{ even} \\
  n^3, & n \text{ odd}
\end{cases} \]

and area C admits

\[ f(n) = \begin{cases} 
  n, & n \text{ even} \\
  n^3, & n \text{ odd}
\end{cases} \]

The algorithms we will consider rarely lie in the A, B, or C parts of the diagram.
The simplest test for \( f \in \Theta(g) \) is this lemma.

**Lemma:** If \( \lim_{n \to \infty} f(n)/g(n) = L \) and \( 0 < L < \infty \), then \( f \in \Theta(g) \).

**Proof:** By the definition of limit, if \( f(n)/g(n) \to L \) and \( 0 < L < \infty \), then \( 0 < L/2 \leq f(n)/g(n) \leq 2L < \infty \) for all sufficiently large \( n \). Hence we can use \( K_1 = L/2 \) and \( K_2 = 2L \) as the constants required by the definition of \( \Theta(g) \).

Example. Is \( f(n) = 3n^2 + 4n + 10 \) in \( \Theta(g) \) for \( g = 8n^2 + 9 \)?

\[
\frac{f(n)}{g(n)} = \frac{3n^2 + 4n + 10}{8n^2 + 9} = \frac{3 + 4/n + 10/n^2}{8 + 9/n^2} \to \frac{3}{8}
\]

\( \frac{f(n)}{g(n)} \in (3/16, 3/4) \), for all sufficiently large \( n \)

\[
0 < \frac{3}{16} \leq \frac{f(n)}{g(n)} \leq \frac{3}{4} < \infty, \text{ for all sufficiently large } n
\]

\( f(n) \in \Theta(g(n)) \).
However, we can still have \( f \in \Theta(g) \) even if \( \lim_{n \to \infty} f(n)/g(n) \) does not exist.

Example.

\[
\begin{align*}
g(n) &= n^2 \\
f(n) &= \begin{cases} 
4n^2, & \text{n even} \\
2n^2, & \text{n odd.}
\end{cases}
\end{align*}
\]

In this case, the ratio has no limit; it oscillates between 4 and 2 forever. However, the ratio has upper bound 4 and lower bound 2, which serve as appropriate constants in the definition of \( \Theta(n^2) \). That is,

\[
0 < 2 \leq \frac{f(n)}{g(n)} \leq 4 < \infty,
\]

which places \( f(n) \in \Theta(g(n)) \).
**Definitions:** limit supremum (lim sup) and limit infimum (lim inf).

\[
\lim_{n \to \infty} \sup h(n) = \lim_{n \to \infty} \sup_{m \geq n} h(m) \\
\lim_{n \to \infty} \inf h(n) = \lim_{n \to \infty} \inf_{m \geq n} h(m).
\]

Note that the limits on the right always exist because \( \sup_{m \geq n} h(m) \) is a monotone decreasing function of \( n \).

Why? As the number of competitors is reduced, the supremum over these competitors can only get smaller.

Similarly, \( \inf_{m \geq n} h(m) \) is monotone-increasing.
Theorem: $f \in \Theta(g)$ if and only if

$$0 < \liminf_{n \to \infty} \frac{f(n)}{g(n)} \leq \limsup_{n \to \infty} \frac{f(n)}{g(n)} < \infty.$$ 

Proof: Let $\alpha = \liminf_{n \to \infty} \frac{f(n)}{g(n)} \leq \limsup_{n \to \infty} \frac{f(n)}{g(n)} = \beta$.

Suppose $0 < \alpha \leq \beta < \infty$. Then, there exists $N_1$ such that $n \geq N_1$ implies

$$\left| \left( \sup_{m \geq n} \frac{f(m)}{g(m)} \right) - \beta \right| \leq \frac{\beta}{2}$$

$$\frac{\beta}{2} \leq \sup_{m \geq n} \frac{f(m)}{g(m)} \leq \frac{3\beta}{2}$$

$$\frac{f(n)}{g(n)} \leq \frac{3\beta}{2} < \infty$$

$$f \in O(g).$$

Similarly, there exists $N_2$ such that $n \geq N_2$ implies

$$\left| \left( \inf_{m \geq n} \frac{f(m)}{g(m)} \right) - \alpha \right| \leq \frac{\alpha}{2}$$

$$\frac{\alpha}{2} \leq \inf_{m \geq n} \frac{f(m)}{g(m)} \leq \frac{3\alpha}{2}$$

$$\frac{f(n)}{g(n)} \geq \frac{\alpha}{2} > 0$$

$$f \in \Omega(g).$$

We conclude $f \in \Theta(g)$. 

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Conversely, suppose that \( f \in \Theta(g) \). Then, there exist integer \( N \) and constants \( 0 < K_1 \leq K_2 < \infty \) such that \( n \geq N \) implies

\[
0 < K_1 \leq \frac{f(n)}{g(n)} \leq K_2 < \infty
\]

\[
\sup_{m \geq N} \frac{f(m)}{g(m)} \leq K_2 < \infty
\]

\[
\sup_{m \geq n} \frac{f(m)}{g(m)} \leq K_2 < \infty, \text{ sup decreases as competitors } [N, n) \text{ are removed}
\]

\[
\beta = \limsup_{n \to \infty} \frac{f(n)}{g(n)} = \limsup_{n \to \infty} \frac{f(m)}{g(m)} \leq K_2 < \infty.
\]

Similarly, \( n \geq N \) implies

\[
\inf_{m \geq N} \frac{f(m)}{g(m)} \geq K_1 > 0
\]

\[
\inf_{m \geq n} \frac{f(m)}{g(m)} \geq K_1 > 0, \text{ inf increases as competitors } [N, n) \text{ are removed}
\]

\[
\alpha = \liminf_{n \to \infty} \frac{f(n)}{g(n)} = \liminf_{n \to \infty} \frac{f(m)}{g(m)} \geq K_1 > 0.
\]

We conclude \( 0 < \alpha \leq \beta < \infty \). ■