3. Suppose \( a, b \in \mathbb{Z} \). If \( a^2(b^2 - 2b) \) is odd, then \( a \) and \( b \) are odd.

   Proof: (by contrapositive)

   Suppose it is not the case that \( a \) and \( b \) are odd.

   Then, either \( a \) is even or \( b \) is even.

   Case (a): If \( a \) is even, then for some integer \( x \),
   \[
a = 2x
   \]
   \[
a^2(b^2 - 2b) = 4x^2(b^2 - 2b) = 2[2x^2(b^2 - 2b)],
   \]
   which implies \( a^2(b^2 - 2b) \) is even. So, \( a^2(b^2 - 2b) \) is not odd.

   Case (b): If \( b \) is even,
   \[
b = 2y
   \]
   \[
a^2(b^2 - 2b) = a^2(4y^2 - 4y) = 2[a^2(2y^2 - 2y)],
   \]
   which again implies \( a^2(b^2 - 2b) \) is even. So, \( a^2(b^2 - 2b) \) is not odd.

   In either case, \( a^2(b^2 - 2b) \) is not odd. ■

7. Suppose \( a, b \in \mathbb{Z} \). If both \( ab \) and \( a + b \) are even, then both \( a \) and \( b \) are even.

   Proof: (by contrapositive)

   Suppose it is not the case that \( a \) and \( b \) are even.

   Then, either \( a \) is odd or \( b \) is odd.

   Case (a): If \( a \) is odd, then \( ab \) has the parity of \( b \) and \( a + b \) has parity opposite from \( b \). So, \( ab \) and \( a + b \) have opposite parity.

   Case (b): If \( b \) is odd, then \( ab \) has the parity of \( a \) and \( a + b \) has parity opposite from \( a \). So, \( ab \) and \( a + b \) have opposite parity.

   In either case, \( ab \) and \( a + b \) cannot both be even. ■

11. Suppose \( x, y \in \mathbb{Z} \). If \( x^2(y + 3) \) is even, then \( x \) is even or \( y \) is odd.

   Proof: (by contrapositive)

   Suppose it is not the case that \( x \) is even or \( y \) is odd.

   Then, \( x \) is odd and \( y \) is even.

   \[
x = 2a + 1 \quad \text{and} \quad y = 2b, \text{ for some} \quad a, b \in \mathbb{Z}, \text{ and}
   \]
   \[
x^2(y + 3) = (2a + 1)^2(2b + 3) = 2b(2a + 1)^2 + 2(2a + 1)^2 + (2a + 1)^2 = 2(2a + 1)^2(b + 1) + 4a^2 + 4a + 1
   \]
   \[
   = 2[(2a + 1)^2(b + 2) + 2a^2 + 2a] + 1,
   \]
   which is odd. ■

13. Suppose \( x \in \mathbb{R} \). If \( x^5 + 7x^3 + 5x \geq x^4 + x^2 + 8, \) then \( x \geq 0 \).

   Proof: (by contrapositive)

   Suppose it is not the case that \( x \geq 0 \).

   Then \( x < 0 \), all odd powers of \( x \) are less than zero, and all even powers of \( x \) are greater than zero.

   The left-hand-side of \( x^5 + 7x^3 + 5x \geq x^4 + x^2 + 8 \) is then less than zero, and the right-hand-side is greater than zero.

   That is, \( x^5 + 7x^3 + 5x < x^4 + x^2 + 8 \). ■
17. If \( n \) is odd, then \( 8 | (n^2 - 1) \).
   Proof: (direct)
   Suppose \( n \) is odd.
   Then, for some integer \( x \), \( n = 2x + 1 \).
   \[
   n^2 - 1 = 4x^2 + 4x + 1 - 1 = 4(x^2 + x) = 4 \cdot 2y = 8y
   \]
since \( x^2 + x \) is always even. So, \( 8 | (n^2 - 1) \). 

19. Let \( a, b \in \mathbb{Z} \) and \( n \in \mathbb{N} \). If \( a \equiv b \mod n \) and \( a \equiv c \mod n \), then \( c \equiv b \mod n \).
   Proof: (direct)
   \[a \equiv b \mod n\] and \[a \equiv c \mod n\] imply
   \[
   n \mid (a - b)
   \]
   \[
   n \mid (a - c)
   \]
   \[
   a - b = xn
   \]
   \[
   a - c = yn
   \]
   \[
   (a - b) - (a - c) = xn - yn = (x - y)n
   \]
   \[
   c - b = (x - y)n,
   \]
   which implies \( c \equiv b \). 

23. Let \( a, b, c \in \mathbb{Z} \) and \( n \in \mathbb{N} \). If \( a \equiv b \mod n \), then \( ca \equiv cb \mod n \).
   Proof: (direct)
   \( a \equiv b \mod n \) implies
   \[
   a - b = xn
   \]
   \[
   ca - cb = c(xn),
   \]
   which implies \( ca \equiv cb \mod n \). 

27. If \( a \equiv 0 \mod 4 \) or \( a \equiv 1 \mod 4 \), then \( \binom{a}{2} \) is even.
   Proof: (direct)
   If \( a \equiv 0 \mod 4 \),
   \[
   \binom{a}{2} = \frac{a(a - 1)}{2} = \frac{4x(4x - 1)}{2} = 2x(4x - 1), \text{ which is even.}
   \]
   If \( a \equiv 1 \mod 4 \),
   \[
   a - 1 = 4x
   \]
   \[
   a = 4x + 1
   \]
   \[
   \binom{a}{2} = \frac{a(a - 1)}{2} = \frac{(4x + 1)4x}{2} = 2x(4x + 1), \text{ which is also even.}
   \]

31. Suppose the division algorithm, applied to \( a \) and \( b \), yields \( a = qb + r \). Then \( \gcd(a, b) = \gcd(r, b) \).
   Proof: (direct) Let \( d = \gcd(a, b) \).
   Then \( d \mid b \) and \( d \mid a \).
   Since \( r = a - qb \), we have \( d \mid r \). So, \( d \) divides both \( b \) and \( r \) and is therefore a competitor for \( \gcd(r, b) \). That is, \( \gcd(r, b) \geq \gcd(a, b) \).
   Now let \( d' = \gcd(r, b) \). Then \( d' \mid r \) and \( d' \mid b \).
   Since \( a = qb + r \), it follows that \( d' \mid a \). So \( d' \) divides both \( a \) and \( b \) and is therefore a competitor for \( \gcd(a, b) \). That is, \( \gcd(a, b) \geq \gcd(r, b) \).
   Combining the inequalities, we have \( \gcd(a, b) = \gcd(r, b) \).