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3. If \( a \) is an odd integer, then \( a^2 + 3a + 5 \) is odd.
   Proof: Let \( a \) be an odd integer.
   \[
   a = 2x + 1, \text{ for some integer } x
   \]
   \[
   a^2 + 3a + 5 = (2x + 1)^2 + 3(2x + 1) + 5 = 4x^2 + 4x + 1 + 6x + 3 + 5 = 4x^2 + 10x + 9 = 2(2x^2 + 5x + 4) + 1
   \]
   \[
   a^2 + 3a + 5 = 2b + 1, \text{ for } b = 2x^2 + 4x + 4.
   \]

   \( a^2 + 3a + 5 \) is odd. \( \blacksquare \)

5. Suppose \( x, y \in \mathbb{Z} \). If \( x \) is even, then \( xy \) is even.
   Proof: Let \( x, y \in \mathbb{Z} \) with \( x \) even.
   \[
   x = 2z, \text{ for some integer } z
   \]
   \[
   xy = (2z)y = 2(zy).
   \]

   Therefore \( xy \) is even. \( \blacksquare \)

9. Suppose \( a \) is an integer. If \( 7 | (4a) \), then \( 7 | a \).
   Proof. Let \( a \) be an integer such that \( 7 | (4a) \).
   Then, there exists integer \( b \) such that \( 4a = 7b \).
   The left side \( 4a \) is even. Therefore the right side \( 7b \) is even.
   Since the product of two odd integers is odd, \( b \) must be even.
   There exists integer \( c \) such that \( b = 2c \), which gives \( 4a = 7b = 14c \), or equivalently \( 2a = 7c \).
   Again, the left side \( 2a \) is even, so \( 7c \) must be even.
   But, the product of two odds is odd, so \( c \) is even.
   So, there exists integer \( d \) such that \( c = 2d \).
   That \( 2a = 7c = 7(2d) = 14d \) or \( a = 7d \).
   Therefore \( 7 | a \). \( \blacksquare \)

13. Suppose \( x, y \in \mathbb{R} \). If \( x^2 + 5y = y^2 + 5x \), then either \( x = y \) or \( x + y = 5 \).
   Proof: Let \( x, y \in \mathbb{R} \) such that \( x^2 + 5y = y^2 + 5x \).
   Case(a) \( x = y \).
   In this case, we have \((x = y) \lor (x + y = 5)\) is true.
   Case (b) \( x \neq y \). Then, we manipulate the equation as follows.
   \[
   x^2 + 5y = y^2 + 5x
   \]
   \[
   x^2 - y^2 = 5(x - y)
   \]
   \[
   (x + y)(x - y) = 5(x - y)
   \]
   \[
   x + y = 5,
   \]
   the division by \( x - y \) being justified by \( x - y \neq 0 \) since in this case, \( x \neq y \).
   Consequently, we again have \((x = y) \lor (x + y = 5)\) is true. \( \blacksquare \)
15. If \( n \in \mathbb{Z} \), then \( n^2 + 3n + 4 \) is even.

   **Proof.** Case 1: \( n \) is even. That is, \( n = 2a \) for some integer \( a \).
   \[
   n^2 + 3n + 4 = 4a^2 + 6a + 4 = 2(2a^2 + 3a + 2),
   \]
   which implies \( n^2 + 3n + 4 \) is even.
   
   Case 2: \( n \) is odd. That is, \( n = 2a + 1 \) for some integer \( a \).
   \[
   n^2 + 3n + 4 = (2a + 1)^2 + 3(2a + 1) + 4 = 4a^2 + 4a + 1 + 6a + 3 + 4 = 4a^2 + 8a + 8 = 2(2a^2 + 4a + 4),
   \]
   which implies \( n^2 + 3n + 4 \) is even. □

19. Suppose \( a, b, c \) are integers. If \( a^2 \mid b \) and \( b^3 \mid c \), then \( a^6 \mid c \).

   **Proof:** There exist integers \( x, y \) such that \( a^2 x = b \) and \( b^3 y = c \). Then,
   \[
   b^3 = (a^2 x)^3 = a^6 x^3,
   \]
   \[
   c = b^3 y = a^6 x^3 y,
   \]
   which implies \( a^6 \mid c \). □

23. If \( n \in \mathbb{N} \), then \( \binom{2n}{n} \) is even.

   **Proof:** By the binomial theorem,
   \[
   \sum_{k=0}^{2n} \binom{2n}{k} = \sum_{k=0}^{2n} \binom{2n}{k} 1^k \cdot 1^{2n-k} = (1 + 1)^{2n} = 2^{2n}.
   \]
   Also,
   \[
   \binom{2n}{0} = \binom{2n}{2n},
   \]
   \[
   \binom{2n}{1} = \binom{2n}{2n-1},
   \]
   \[
   \binom{2n}{2} = \binom{2n}{2n-2},
   \]
   \[
   \vdots = \vdots
   \]
   \[
   \binom{2n}{n-1} = \binom{2n}{n+1}
   \]
   So,
   \[
   2^{2n} = \binom{2n}{n} + 2 \sum_{k=0}^{n-1} \binom{2n}{k}
   \]
   \[
   \binom{2n}{n} = 2^{2n} - 2 \sum_{k=0}^{n-1} \binom{2n}{k} = 2 \left[ 2^{2n-1} - \sum_{k=0}^{n-1} \binom{2n}{k} \right],
   \]
   which implies \( \binom{2n}{n} \) is even. □

27. Suppose \( a, b \in \mathbb{N} \). If \( \gcd(a, b) > 1 \), then \( b \mid a \) or \( b \) is not prime.

   **Proof.** Suppose \( \gcd(a, b) > 1 \). We consider two cases.
   
   Case 1: \( b \) is not prime. In this case, we are finished, since the conclusion \( (b \mid a) \lor (b \text{ is not prime}) \) is true.
   
   Case 2: \( b \) is prime. In this case, the only divisors of \( b \) are 1 and \( b \). Since the hypothesis excludes 1, we have \( \gcd(a, b) = b \). That is, \( b \mid a \), since the gcd must divide both of its arguments. □