3. How many lists of length 3 can be made from the symbols A, B, C, D, E, F, if . . .

(a) repetition is allowed: \(6^3 = 216\).
(b) repetition is not allowed: \(6 \cdot 5 \cdot 4 = 120\).
(c) repetition is not allowed and the list must contain A: \(3 \cdot 5 \cdot 4 = 60\). Place the A in one of the three positions, then choose from the remaining 5 letters for the other positions.
(d) repetition is allowed and the list must contain A:
\[
\binom{3}{1} \cdot 5^2 + \binom{3}{2} \cdot 5 + \binom{3}{3} = 3(25) + 3(5) + 1 = 91.
\]
List must contain 1 A or 2 A’s or 3 A’s.

7. Considering 8-bit strings, such as 01101100 . . .

(a) How many such strings are there? \(2^8 = 256\).
(b) How many strings end with zero? \(2^7 = 128\).
(c) How many strings have one in positions 2 and 4? \(2^6 = 64\).
(d) How many strings have one in position 2 or 4? \(2^7 + 2^7 - 2^6 = 256 - 64 = 192\) by Inclusion-Exclusion.

9. Considering four-letter codes from A, B, . . ., Z:

(a) How many strings? \(26^4 = 456976\)
(b) How many strings with no two consecutive entries the same? \(26 \cdot 25 \cdot 25 \cdot 25 = 406250\).

5. Using only pencil and paper, find \(120!/118!\).
\[
\frac{120!}{118!} = \frac{120 \cdot 119 \cdot 118!}{118!} = 120 \cdot 119 = (120)^2 - 120 = 14400 - 120 = 14280.
\]

7. Compute \(N\), the count of 9-digit numbers that can be made from 1, 2, . . ., 9 if repetition is not allowed and all the odd digits occur followed by all the even digits.

No repetition with 9-digit numbers means all of the digits must be used. There are five odds and four evens. So
\[
N = (5 \cdot 4 \cdot 3 \cdot 2 \cdot 1)(4 \cdot 3 \cdot 2 \cdot 1) = 5! \cdot 4! = 120(24) = 1440(2) = 2880.
\]

5. How many 16-bit binary strings contain exactly seven ones?

\[
\binom{16}{7} = \frac{16!}{7! \cdot 9!} = \frac{16 \cdot 15 \cdot 14 \cdot 13 \cdot 12 \cdot 11 \cdot 10}{7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2} = \frac{16 \cdot 14 \cdot 13 \cdot 12 \cdot 11 \cdot 10}{7 \cdot 6 \cdot 4 \cdot 2} = \frac{16 \cdot 13 \cdot 12 \cdot 11 \cdot 10}{6 \cdot 4} = 8 \cdot 13 \cdot 11 \cdot 10
\]
\[
= 880 \cdot 13 = 11440.
\]

7. \(|\{X \in P(\{0,1,\ldots,9\}) : |X| < 4\}| = ?
\[
|\{X \in P(\{0,1,\ldots,9\}) : |X| < 4\}| = \sum_{j=0}^{3} \binom{10}{j} = 1 + 10 + \frac{10 \cdot 9}{2} + \frac{10 \cdot 9 \cdot 8}{2 \cdot 3} = 56 + 120 = 176.
\]
9. Consider strings of length 6 made from letters A, B, C, D, E, F with no repetition. How many such strings have the D occurring before the A.
   Choose positions for the A and D in \( \binom{6}{2} = 15 \) ways. Fill the remaining 4 positions in 4! = 24 ways. The total is 15 \cdot 24 = 360 ways.

11. How many positive 10-digit integers contain no zeros and exactly three sixes?
   \[
   \binom{10}{3} \cdot 8^7 = \frac{10 \cdot 9 \cdot 8}{2 \cdot 3} \cdot 8^7 = 120 \cdot 8^7 = 251658240.
   \]

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3. Use the binomial theorem to find the coefficient \( c_8 \) of \( x^8 \) in \( (x + 2)^{13} \).
   \[
   c_8 = \binom{13}{8} \cdot 2^{13-8} = 32 \cdot \binom{13}{5} = 32 \cdot \frac{13 \cdot 12 \cdot 11 \cdot 10 \cdot 9}{2 \cdot 3 \cdot 4 \cdot 5} = 32 \cdot 13 \cdot 11 \cdot 9 = 41184.
   \]

5. Use the binomial theorem to prove \( \sum_{k=0}^{n} \binom{n}{k} = 2^n \).
   \[
   \sum_{k=0}^{n} \binom{n}{k} = \sum_{k=0}^{n} \binom{n}{k} 1^{k} 1^{n-k} = (1 + 1)^n = 2^n.
   \]

7. Use the binomial theorem to prove \( \sum_{k=0}^{n} 3^k \binom{n}{k} = 4^n \).
   \[
   \sum_{k=0}^{n} 3^k \binom{n}{k} = \sum_{k=0}^{n} \binom{n}{k} 3^{k} 1^{n-k} = (3 + 1)^n = 4^n.
   \]

9. Use the binomial theorem to show \( S = \binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \binom{n}{3} + \ldots + (-1)^n \binom{n}{n} = 0 \).
   \[
   S = \sum_{j=0}^{n} \binom{n}{j} (-1)^j = \sum_{j=0}^{n} \binom{n}{j} (-1)^j (1)^{n-j} = [1 + (-1)]^n = 0^n = 0.
   \]

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1. 523 students are history majors, math majors, or both. 100 students are math majors. 33 students are double majors: math and history. How many students are history majors?
   Let \( A \) be the set of history majors; let \( B \) be the set of math majors. We have \( |A \cup B| = 523 \), \( |B| = 100 \), and \( |A \cap B| = 33 \).
   By inclusion-exclusion, we have \( |A \cup B| = |A| + |B| - |A \cap B| \).
   Consequently, 523 = \( |A| + 100 - 33 \), or \( |A| = 523 - 67 = 456 \).

3. How many 4-digit positive integers are either even or contain no zeros. We assume 4-digit constructions beginning with one or more zeros are not 4-digit integers.
   Let \( A \) be the evens; let \( B \) be those containing no zeros. We want \( |A \cup B| \).
   Since an integer is even if and only if its last digit is even, we have \( |A| = 9 \cdot 10 \cdot 10 \cdot 5 = 4500 \), since there are 5 even digits
   Also, \( |B| = 9^4 = 6561 \), and \( |A \cap B| = 9 \cdot 9 \cdot 9 \cdot 4 = 2916 \), since there are 4 even nonzero digits.
   So, \( |A \cup B| = 4500 + 6561 - 2916 = 8145 \).

7. Using a standard 52-card deck, how many 4-card sets have either (a) all four cards in the same suit or (b) all four cards are red?
   Let \( A \) be the 4-card sets with all four in the same suit. Let \( B \) be the 4-card sets in which all are red. We construct an element of \( A \) by first choosing the suit and then four cards from that suit.
   \[
   |A| = \binom{4}{1} \cdot \binom{13}{4} = 4 \cdot \frac{13 \cdot 12 \cdot 11 \cdot 10}{2 \cdot 3 \cdot 4} = 26 \cdot 110 = 2860
   \]
   \[
   |B| = \binom{26}{4} = \frac{26 \cdot 25 \cdot 24 \cdot 23}{4 \cdot 3 \cdot 2} = 26 \cdot 25 \cdot 23 = 14950,
   \]
the last because there are 26 red cards in the deck. For \( A \cap B \), we first choose one of the two red suits and then four cards from that suit:

\[
|A \cap B| = \binom{2}{1} \cdot \binom{13}{4} = 2 \cdot \frac{13 \cdot 12 \cdot 11 \cdot 10}{2 \cdot 3 \cdot 4} = 13 \cdot 11 \cdot 10 = 1430
\]

\[
|A \cup B| = |A| + |B| - |A \cap B| = 2860 + 14950 - 1430 = 16380,
\]

the last by inclusion-exclusion.

9. A 4-letter list is made from the letters L, I, S, T, E, D as follows: (a) repetition is allowed and (b) the first two letters are vowels or the list ends in D. How many lists?

Let \( A \) be the set of such lists, given that repetition is allowed and the first two letters are vowels. Let \( B \) be the set of such lists, given that repetition is allowed and the last letter is D. We want \( |A \cup B| \).

Since the choices include only two vowels, I and E, we have

\[
|A| = 2 \cdot 2 \cdot 6 \cdot 6 = 144
\]

\[
|B| = 6 \cdot 6 \cdot 6 \cdot 1 = 6^3 = 216,
\]

the latter since the final letter must be D for membership in B. Also, if conditions (a) and (b) are imposed,

\[
|A \cap B| = 2 \cdot 2 \cdot 6 \cdot 1 = 24
\]

\[
|A \cup B| = |A| + |B| - |A \cap B| = 144 + 216 - 24 = 336.
\]