3. Show the four distinct functions mapping \( \{a, b\} \) to \( \{0, 1\} \).

<table>
<thead>
<tr>
<th>( f_1 )</th>
<th>( f_2 )</th>
<th>( f_3 )</th>
<th>( f_4 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( a )</td>
<td>0</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>( b )</td>
<td>0</td>
<td>1</td>
<td>0</td>
</tr>
</tbody>
</table>

4. Show the eight distinct functions mapping \( \{a, b, c\} \) to \( \{0, 1\} \).

<table>
<thead>
<tr>
<th>( f_1 )</th>
<th>( f_2 )</th>
<th>( f_3 )</th>
<th>( f_4 )</th>
<th>( f_5 )</th>
<th>( f_6 )</th>
<th>( f_7 )</th>
<th>( f_8 )</th>
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<td>( c )</td>
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<td>0</td>
<td>1</td>
<td>0</td>
</tr>
</tbody>
</table>

5. Give an example of a relation from \( \{a, b, c, d\} \) to \( \{d, e\} \) that is not a function.

\[ R = \{(a, d), (a, e)\} \text{ fails to be a function since (1) it maps } a \text{ to two different values and (2) it fails to map } b, c, d. \]

9. Consider the set \( R = \{(x^2, x) : x \in \mathbb{R}\} \). Does this relation constitute a function mapping \( \mathbb{R} \) to \( \mathbb{R} \)?

No. \( R \) contains the pairs \((1, 1)\) and \((1, -1)\), which maps 1 to two distinct values. Also, a negative entries in \( R \), such as \(-5\), do not appear in the left position of any pair.

p. 202

1. Let \( A = \{1, 2, 3, 4\}, B = \{a, b, c\} \). Give an example of a function that is neither injective nor surjective.

Let \( f(1) = f(2) = f(3) = f(4) = a \). \( f \) is not injective, since 1 and 2 map to the same target. \( f \) is not surjective, since \( b \) and \( c \) are excluded from the targets.

9. Let \( f : \mathbb{R} - \{2\} \to \mathbb{R} - \{5\} \text{ via } f(x) = \frac{(5x+1)}{(x-2)}. \) Prove that \( f \) is bijective.

Proof: \( f \) is well-defined since 2 is excluded from the domain and therefore \( x - 2 \neq 0 \) for all \( x \) in the domain.

Also, if \( f(x) = 5 \) for some \( x \neq 2 \), then

\[
\frac{5x+1}{x-2} = 5 \\
5x+1 = 5(x-2) = 5x-10 \\
1 = -10,
\]

a contradiction. So \( f : \mathbb{R} - \{2\} \to \mathbb{R} - \{5\} \).

If \( f(x) = f(y) \), for some \( x \neq 2 \) and \( y \neq 2 \),

\[
\frac{5x+1}{x-2} = \frac{5y+1}{y-2} \\
(5x+1)(y-2) = (5y+1)(x-2) \\
5xy - 10x + y - 2 = 5xy - 10y + x - 2 \\
11y = 11x \\
x = y.
\]
So, \( f \) is injective. Also, for any \( y \neq 5 \), we investigate \( f(x) = y \) to determine if there exists an \( x \) such that \( f \) maps \( x \) to \( y \).

\[
\frac{5x + 1}{x - 2} = y
\]

\[
5x + 1 = y(x - 2) = xy - 2y
\]

\[
5x - xy = -2y - 1
\]

\[
x(5 - y) = -2y - 1
\]

\[
x = \frac{-2y - 1}{5 - y}.
\]

So, \( f \) is surjective. Being both injective and surjective, \( f \) is bijective.

13. Let \( f : \mathbb{R}^2 \to \mathbb{R}^2 \) via \( f(x, y) = (xy, x^3) \). Is \( f \) injective? surjective? bijective?

We note that \( f(0, y) = (0, 0) \) for any \( y \). Consequently, \( f \) is not injective.

Also, if \( f(x, y) = (1, 0) \), we have

\[
(xy, x^3) = (1, 0)
\]

\[
x^3 = 0
\]

\[
x = 0
\]

\[
xy = 0,
\]

which contradicts \( xy = 1 \). So, \( f \) is not surjective.

As bijective requires both injective and surjective, \( f \) is not bijective.

15. Consider functions mapping \( \{A, B, C, D, E, F, G\} \) to \( \{1, 2, 3, 4, 5, 6, 7\} \).

(a) How many such functions? \( 7^7 = 823543 \), as we have 7 possible targets for \( A \), 7 for \( B \), and so forth.

(b) How many such functions are injective? \( 7! = 5040 \), as we have 7 possible targets for \( A \), 6 for \( B \), and so forth.

(c) How many such functions are surjective? \( 7! = 5040 \), since we have 7 possible sources for target 1, then 6 possible sources for target 2, and so forth.

Indeed, to be injective, we need 7 different targets for the 7 sources, since we can’t use a target twice. Thus an injection is also a surjection and vice versa.

(d) How many such functions are bijective? \( 7! \), the same 7! functions chosen in (b) and (c).

p. 205

1. If six integers are chosen at random, then at least two of them will have the same remainder when divided by 5. That is, at least two of them will be equal mod 5.

Proof: Let the chosen integers be \( A = \{x_1, x_2, \ldots, x_6\} \). Let \( B = \{0, 1, 2, 3, 4\} \). Let \( f : A \to B \) via \( f(x) = x \mod 5 \).

We are mapping six integers into a set containing only 5 integers. By the pigeonhole principle, at least two sources must map to the same target. That is, there exist \( i \neq j \) with \( x_i \mod 5 = x_j \mod 5 \).
3. If six natural numbers are chosen at random, then the sum or the difference of two of them is divisible by 9.

Proof: Let the numbers be $A = \{x_1, x_2, \ldots, x_6\}$. If there exist $i \neq j$ with $x_i \equiv x_j \mod 9$, then $x_i - x_j = 0 \mod 9$, which implies $9|(x_i - x_j)$, and we are finished. Otherwise, the integers must all be distinct, and we must have $i \neq j \Rightarrow x_i \neq x_j \mod 9$. In this case, let $B = \{0, 1, 2, \ldots, 8\}$, and let $f : A \to B$ via $f(x_i) = x_i \mod 9$, yielding 6 distinct $f(x_i)$ values in the 9-member set $B$. Consequently, at least 5 distinct $f(x_i)$ values lie in the 8-member set $B' = \{1, 2, \ldots, 8\}$.

If these 5 distinct $f(x_i)$ values include any of the pairs $\{1, 8\}, \{2, 7\}, \{3, 6\}, \text{or} \{4, 5\}$, then the corresponding $x_i + x_j \equiv 0 \mod 9$. So, to avoid an $x_i + x_j \equiv 0 \mod 9$ situation, we need to choose each of the five distinct $f(x_i)$ values from a different pair.

Because of the excluded pairs, each choice of an $f(x_i)$ value precludes both that value and its partner in one of the 4 excluded pairs above.

For example, if we choose 1, we knock out both 1 and 8 from further choices. If we choose 2, we knock out both 2 and 7, and so forth.

Consequently, after choosing four of the five distinct values, we have excluded all eight values from the field $\{1, 2, 3, 4, 5, 6, 7, 8\}$, and we can’t make our fifth choice without creating one of the excluded pairs.

Therefore, at least two of the five distinct $f(x_i)$ must lie in an excluded pair, and, for those two, say $x_i$ and $x_j$, we have $x_i + x_j \equiv 0 \mod 9$.

p. 208

3. Let $A = \{1, 2, 3\}$. Let $f : A \to A$ be $f = \{(1, 2), (2, 2), (3, 1)\}$ and $g : A \to A$ be $g = \{(1, 3), (2, 1), (3, 2)\}$. Find $g \circ f$ and $f \circ g$.

\[
\begin{align*}
(g \circ f)(1) &= g(f(1)) = g(2) = 1 \\
(g \circ f)(2) &= g(f(2)) = g(2) = 1 \\
(g \circ f)(3) &= g(f(3)) = g(1) = 3.
\end{align*}
\]

So, $g \circ f = \{(1, 1), (2, 1), (3, 3)\}$.

\[
\begin{align*}
(f \circ g)(1) &= f(g(1)) = f(3) = 1 \\
(f \circ g)(2) &= f(g(2)) = f(1) = 2 \\
(f \circ g)(3) &= f(g(3)) = f(2) = 2.
\end{align*}
\]

So, $f \circ g = \{(1, 1), (2, 2), (3, 2)\}$.

9. $f : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$ and $g : \mathbb{Z} \to \mathbb{Z} \times \mathbb{Z}$ via

\[
\begin{align*}
f(m, n) &= m + n \\
g(m) &= (m, m).
\end{align*}
\]

Find formulas for $g \circ f$ and $f \circ g$.

We have $g \circ f : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z} \times \mathbb{Z}$ via

\[
(g \circ f)(m, n) = g(m + n) = (m + n, m + n).
\]

Also, $f \circ g : \mathbb{Z} \to \mathbb{Z}$ via

\[
(f \circ g)(n) = f(n, n) = n + n = 2n.
\]

p. 212

3. Let $B = \{2^n : n \in \mathbb{Z}\} = \{\ldots, 1/8, 1/4, 1/2, 1, 2, 4, 8, \ldots\}$. Show that $f : \mathbb{Z} \to B$, via $f(k) = 2^k$, is bijective and find $f^{-1}$.

We have $x \in B$ implies $x = 2^k$ for some $k \in \mathbb{Z}$. Then $f(k) = 2^k = x$, which shows that $f$ is surjective.

Also, $f(j) = f(k)$ implies $2^j = 2^k$, which implies $j = \log_2(2^j) = \log_2(2^k) = k$. So, $f$ is injective. Being both surjective and injective, $f$ is bijective.
For the inverse, we solve $f(x) = y$ for $x$.

\[
\begin{align*}
  y &= f(x) = 2^x \\
  \log_2 y &= x.
\end{align*}
\]

So, $f^{-1}(y) = \log_2 y$, which we check as follows.

\[
\begin{align*}
  f(f^{-1}(y)) &= f(\log_2 y) = 2^{\log_2 y} = y \\
  f^{-1}(f(x)) &= f^{-1}(2^x) = \log_2(2^x) = x.
\end{align*}
\]

7. Show that the function $f : \mathbb{R}^2 \to \mathbb{R}^2$ via $f(x, y) = ((x^2 + 1)y, x^3)$ is bijective. Then find its inverse.

To show that $f$ is surjective, let $(u, v) \in \mathbb{R}^2$. We seek $(x, y)$ such that $f(x, y) = (u, v)$. That is,

\[
\begin{align*}
  u &= (x^2 + 1)y \\
  v &= x^3 \\
  x &= v^{1/3} \\
  y &= \frac{u}{x^2 + 1} = \frac{u}{v^{2/3} + 1} \\
  (x, y) &= \left( v^{1/3}, \frac{u}{v^{2/3} + 1} \right).
\end{align*}
\]

We check via

\[
\begin{align*}
  f(x, y) &= f \left( v^{1/3}, \frac{u}{v^{2/3} + 1} \right) = \left( (v^{2/3} + 1) \cdot \frac{u}{v^{2/3} + 1}, v \right) = (u, v)
\end{align*}
\]

So, $f$ is surjective.

To show the $f$ is injective, suppose $f(x, y) = f(u, v)$. Then

\[
\begin{align*}
  ((x^2 + 1)y, x^3) &= ((u^2 + 1)v, u^3) \\
  x &= u \\
  (x^2 + 1)y &= (u^2 + 1)v \\
  (u^2 + 1)y &= (u^2 + 1)v \\
  y &= v.
\end{align*}
\]

Hence $(x, y) = (u, v)$, and $f$ is injective. Being both surjective and injective, $f$ is bijective.

To solve for $f^{-1}$, we solve $f(x, y) = (u, v)$ for $(x, y)$.

\[
\begin{align*}
  (u, v) &= f(x, y) = ((x^2 + 1)y, x^3) \\
  v &= x^3 \\
  x &= v^{1/3} \\
  u &= (x^2 + 1)y \\
  y &= \frac{u}{x^2 + 1} = \frac{u}{v^{2/3} + 1} \\
  f^{-1}(u, v) &= \left( v^{1/3}, \frac{u}{v^{2/3} + 1} \right).
\end{align*}
\]

9. Let $f : \mathbb{R} \times \mathcal{N} \to \mathcal{N} \times \mathbb{R}$ via $f(x, y) = (y, 3xy)$. Check that $f$ is bijective and find its inverse.
Let \((k, x) \in \mathcal{N} \times \mathcal{R}\). We seek \((y, j) \in \mathcal{R} \times \mathcal{N}\) such that \(f(y, j) = (k, x)\).

\[
\begin{align*}
f(y, j) &= (j, 3jy) = (k, x) \\
j &= k \\
3jy &= x \\
y &= \frac{x}{3j}
\end{align*}
\]

\[
f(y, j) = f \left( \frac{x}{3j}, j \right) = (j, 3j \cdot \frac{x}{3j}) = (k, x).
\]

So, \(f\) is surjective. Now, suppose \(f(x, j) = f(y, k)\).

\[
\begin{align*}
(j, 3jx) &= (k, 3ky) \\
j &= k \\
3jx &= 3ky = 3jy \\
x &= y \\
(x, j) &= (y, k).
\end{align*}
\]

So, \(f\) is injective and therefore bijective. For the inverse, we solve \(f(x, j) = (k, y)\) for \((x, j)\).

\[
\begin{align*}
f(x, j) &= (j, 3jx) = (k, y) \\
j &= k \\
3jx &= y \\
3kx &= y \\
x &= \frac{y}{3k} \\
f^{-1}(y, k) &= (x, j) = \left( \frac{y}{3k}, k \right).
\end{align*}
\]

p. 214

3. Consider functions \(f : \{1, 2, 3, 4, 5, 6, 7\} \to \{0, 1, 2, 3, 4\}\). How many such functions have the property that \(|f^{-1}\{\{3\}\}| = 3\)?

We have \(\binom{7}{3}\) ways of choosing the three domain elements that will map to 3. The remaining four domain elements must map to \(\{0, 1, 2, 4\}\), which can be done in \(4^4\) ways. The total number of functions is then

\[
\binom{7}{3} \cdot 4^4 = 2^8 \cdot \binom{7}{3} = 256 \cdot \frac{7 \cdot 6 \cdot 5}{1 \cdot 2 \cdot 3} = 256 \cdot 35 = 8960.
\]

5. Let \(f : A \to B\). For \(X \subseteq A\), show that \(X \subseteq f^{-1}(f(X))\).

Proof: Let \(x \in X\), which implies \(f(x) \in f(X)\). Then

\[
f^{-1}(f(X)) = \{y \in A : f(y) \in f(X)\}
\]

\[
x \in f^{-1}(f(X)). \quad \Box
\]

9. Given \(f : A \to B\) and subsets \(W, X \subseteq A\), prove that \(f(W \cup X) = f(W) \cup f(X)\).

Proof: Let \(y \in f(W \cup X)\). Then there exists \(x \in W \cup X\) such that \(f(x) = y\).

Case (a) : \(x \in W \Rightarrow y = f(x) \in f(W) \subseteq f(W) \cup f(X)\)

\[
y \in f(W) \cup f(X)
\]

Case (b) : \(x \in X \Rightarrow y = f(x) \in f(X) \subseteq f(W) \cup f(X)\)

\[
y \in f(W) \cup f(X)
\]

In either case \(y \in f(W) \cup f(X)\). Therefore \(f(W \cup X) \subseteq f(W) \cup f(X)\).
Now let $y \in f(W) \cup f(X)$.

Case (a): $y \in f(W) \implies \exists x \in W$ with $f(x) = y \implies x \in W \cup X$ with $f(x) = y \implies y \in f(W \cup X)$

Case (b): $y \in f(X) \implies \exists x \in X$ with $f(x) = y \implies x \in W \cup X$ with $f(x) = y \implies y \in f(W \cup X)$.

In either case $y \in f(W \cup X)$. Therefore $f(W) \cup f(X) \subseteq f(W \cup X)$. We conclude $f(W \cup X) = f(W) \cup f(X)$. \hfill\qed