3. For every \( n \in \mathbb{N} \), \( \sum_{k=1}^{n} k^2 = n(n+1)(2n+1)/6 \).

Proof: (Induction)

Base case: For \( n = 1 \),
\[
\sum_{k=1}^{1} k^2 = 1^2 = 1 \\
\frac{n(n+1)(2n+1)}{6} = \frac{(1)(2)(3)}{6} = 1.
\]

Inductive case: For \( n > 1 \), we assume \( \sum_{k=1}^{n-1} k^2 = (n-1)(n)(2n-1)/6 = n(n-1)(2n-1)/6 \). Then,
\[
\sum_{k=1}^{n} k^2 = \left( \sum_{k=1}^{n-1} k^2 \right) + n^2 = \frac{(n-1)(2n-1) + n^2}{6} = \frac{6n^2 + n(n-1)(2n-1)}{6} \\
= \frac{n(6n + (n-1)(2n-1))}{6} = \frac{n(6n + 2n^2 - 3n + 1)}{6} = \frac{n(2n^2 + 3n + 1)}{6} = \frac{n(n+1)(2n+1)}{6}.
\]

7. For \( n \in \mathbb{N} \), \( \sum_{k=1}^{n} k(k+2) = n(n+1)(2n+7)/6 \).

Proof: Assuming we know \( \sum_{k=0}^{n} k = n(n+1)/2 \) and \( \sum_{k=1}^{n} k^2 = n(n+1)(2n+1)/6 \), we have
\[
\sum_{k=1}^{n} k(k+2) = \sum_{k=1}^{n} k^2 + 2 \sum_{k=1}^{n} k = \frac{n(n+1)(2n+1)}{6} + 2 \cdot \frac{n(n+1)}{2} = n(n+1)(2n+1)/6 + 6n(n+1) \\
= \frac{n(n+1)(2n+1+6)}{6} = \frac{n(n+1)(2n+7)}{6}.
\]

Alternative proof: (Induction)

Base case: For \( n = 1 \),
\[
\sum_{k=1}^{1} k(k+2) = 1(3) = 3 \\
\frac{n(n+1)(2n+7)}{6} = \frac{(1)(2)(9)}{6} = 3.
\]

Inductive case: For \( n > 1 \), we assume \( \sum_{k=1}^{n-1} k(k+2) = (n-1)(n)(2n-1)/6 = n(n-1)(2n+5)/6 \). Then,
\[
\sum_{k=1}^{n} k(k+2) = \left( \sum_{k=1}^{n-1} k(k+2) \right) + n(n+2) = \frac{(n-1)(2n+5)}{6} + n(n+2) = \frac{n(n-1)(2n+5) + 6n(n+2)}{6} \\
= \frac{n(2n^2 + 3n - 5 + 6n + 12)}{6} = \frac{n(2n^2 + 9n + 7)}{6} = \frac{n(n+1)(2n+7)}{6}.
\]

11. For any integer \( n \geq 0 \), \( 3|(n^3 + 5n + 6) \).

Proof: (Induction)

Base case: For \( n = 0 \), we have \( n^3 + 5n + 6 = 6 \), which is divisible by 3.

Inductive case: For \( n > 1 \), assume that \( (n-1)^3 + 5(n-1) + 6 = 3k \) for some integer \( k \). Then
\[
n^3 + 5n + 6 = [(n-1)^3 + 5(n-1) + 6 + 6] = (n-1)^3 + 3(n-1)^2 + 3(n-1) + 1 + 5(n-1) + 5 + 6 \\
= [(n-1)^3 + 5(n-1) + 6] + 3(n-1)^2 + 3(n-1) + 6 = 3k + 3[(n-1)^2 + (n-1) + 2] \\
= 3[k + (n-1)^2 + (n-1) + 2].
\]
We conclude \( 3|(n^3 + 5n + 6) \).
19. Prove that $\sum_{k=1}^{n}(1/k^2) \leq 1 + n/2$.
Proof: (Induction)
Base case: For $n = 1$, $\sum_{k=1}^{1}(1/k^2) = 1 \leq 3/2 = 1 + 1/2$.
Inductive case: For $n > 1$, assume that $\sum_{k=1}^{n-1}(1/k^2) \leq 1 + (n-1)/2$. Then,
\[
\sum_{k=1}^{n} \frac{1}{k^2} = \left(\sum_{k=1}^{n-1} \frac{1}{k^2}\right) + \frac{1}{n^2} \leq 1 + \frac{n-1}{2} + \frac{1}{n^2} = 1 + \frac{n}{2} + \left(\frac{1}{n^2} - \frac{1}{2}\right) \leq 1 + \frac{n}{2},
\]
since $(1/n^2 - 1/2)$ is negative for $n > 1$. ■

24. Prove $\sum_{k=1}^{n} k(n) = n \cdot 2^{n-1}$ for all $n \in \mathcal{N}$.
Proof: (Induction)
Base case: For $n = 1$, $\sum_{k=1}^{1} k(n) = 1(1) = 1 = 1 \cdot 2^0 = 1 \cdot 2^{1-1}$.
Inductive case: For $n > 0$, we assume that $\sum_{k=1}^{n-1} k(n-1) = (n-1) \cdot 2^{n-2}$. Then,
\[
\sum_{k=1}^{n} k(n) = \left[\sum_{k=1}^{n-1} k(n)\right] + n(n) = n + \sum_{k=1}^{n-1} k(n-1) + n = n + (n-1) \cdot 2^{n-2} + \sum_{k=0}^{n-2} (k+1)(n-1)
\]
\[
= n + (n-1) \cdot 2^{n-2} + \sum_{k=1}^{n-1} k(n-1) - (n-1) \cdot 2^{n-2} + \sum_{k=0}^{n-1} (n-1) - (n-1)
\]
\[
= n + 2(n-1) \cdot 2^{n-2} - n + 2^{n-1} = (n-1)2^{n-1} + 2^{n-1} = 2^{n-1}[1 + (n-1)] = n \cdot 2^{n-1}. ■
\]

27. Concerning the Fibonacci sequence, $F_n$, prove that $\sum_{k=1}^{n} F_{2k-1} = F_n$.
Proof: (Induction)
Recall that $F_1 = 1, F_2 = 1$, and $F_n = F_{n-1} + F_{n-2}$ for $n > 2$.
Base case: For $n = 1$, $\sum_{k=1}^{n} F_{2k-1} = F_{2(1)-1} = F_1 = 1 = F_2 = F_{2(1)} = F_2$.
Inductive case: For $n > 1$, we assume $\sum_{k=1}^{n-1} F_{2k-1} = F_{2(n-1)}$. Then,
\[
\sum_{k=1}^{n} F_{2k-1} = \left(\sum_{k=1}^{n-1} F_{2k-1}\right) + F_{2n-1} = F_{2(n-1)} + F_{2n-1} = F_{2n-2} + F_{2n-1} = F_{2n}. ■
\]